

GENERALIZED OHM'S LAW IN MULTIFLUID SYSTEMS

by

Shi-Tsan Wu

FACILITY FORM 602	N 68-30283	
	(ACCESSION NUMBER)	(THRU)
	60	1
	(PAGES)	(CODE)
	00-5588	25
	(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

This research work was supported by
the National Aeronautics and Space Administration
under research grant (NsG-381.)

NRL-01-002-001

University of Alabama Research Institute
Huntsville, Alabama

May, 1968



UARI Research Report No. 49

GENERALIZED OHM'S LAW IN MULTIFLUID SYSTEMS

by

Shi-Tsan Wu

This research work was supported by
the National Aeronautics and Space Administration
under research grant (NsG-381.)

NB L-01-002-001

University of Alabama Research Institute
Huntsville, Alabama

May, 1968

GENERALIZED OHM'S LAW IN MULTIFLUID SYSTEMS

SUMMARY

A generalized Ohm's law in multifluid systems under non-Local-Thermodynamic-Equilibrium (non-LTE) is derived through the macroscopic description of the Boltzmann equation. It is shown that the first order approximation can be written as a function of the electric and magnetic field. The second order approximation is shown to be a function of the electric and magnetic field with the effects of the gradients of density, temperature, and mean flow velocity of individual mixture components, and of the effects of the internal degrees of freedom of the particles. The derivation is limited to a slightly ionized gas. Some applications in the area of astrophysical problems are discussed.

ACKNOWLEDGEMENTS

The author is indebted to Dr. Yoshinari Nakagawa, High Altitude Observatory of National Center for Atmospheric Research for suggestions to investigate this problem, and to Dr. J. J. Brainerd for reading this manuscript.

The author also wishes to express his thanks to Mrs. Ann White for typing this paper.

CONTENTS

Summary

- I. Introduction
- II. Basic Equations
- III. Method of Approximation
- IV. Ohm's Law for N-Component Gas Mixture under non-LTE in the First Order Approximation
- V. Ohm's Law for N-Component Gas Mixture under non-LTE in the Second Order Approximation
- VI. Calculation of the Kinetic Coefficients
- VII. The Electric Conductivity and the Generalized Ohm's Law for a Partially Ionized Plasma Under Non-LTE.
- VIII. Discussions

I. INTRODUCTION

Applied research in plasma containment, heating, or acceleration, and research in magnetogasdynamics presently use simplified versions of the generalized Ohm's law to describe conduction of current through a high-temperature gas mixture. Important effects are usually neglected; examples are: (1) the electromotive forces due to temperature and density gradients, (2) the contributions to the transport coefficients by components other than the electrons, neutral or singly-ionized atoms of a single species, and by thermal diffusion, and (3) the contribution of the internal degrees of freedom of the particle. Therefore, this paper is devoted to the formulation of a conduction (Ohm's) law of general validity for a multifluid system under conditions of non-local-thermodynamic-equilibrium (non-LTE), and to determine the functional dependence between the current density and the acting force mechanisms, (i.e., electric field, magnetic field and the gradients of the macroscopic quantities).

This investigation of Ohm's law is based on the derivation of a closed set of transport equations for the mean velocities of the N-component gas mixture under non-LTE conditions by means of the macroscopic description of the Boltzmann equation. In order to allow each component gas in the mixture to be at its own temperature and mean velocity (i.e., at macroscopic non-LTE), we use the velocity distribution developed by Wu [1], which refers each component gas to its own system center of mass. Application of the Chapman-Enskog [2] method of successive approximations, together with the chosen distribution function, enables us to obtain directly, for an N-component gas mixture under non-LTE, both the form of Ohm's

law and expressions for its coefficients.

Recently, Demetriades and Argyropoulos [3] have derived the Ohm's law based on the 13-moments method originated by Grad [4], and on the collision model given by Aligevskiy and Zhdanov [5] for a nonisothermal plasma (with temperature and pressure gradients but ignoring the internal degrees of freedom). The results presented in this study are similar to those obtained by Demetriades and Argyropoulos but have new features due to the contribution of the internal degrees of freedom of the particle. Further, without elaborate calculation, the results show that the electromagnetic field will affect the magnitude of the transport phenomena.

II. BASIC EQUATIONS

In the kinetic description of an N-component gas mixture, the distribution function $f_s(\mathbf{v}_s, \mathbf{r}, t)$ for each constituent gas and the corresponding Boltzmann equation are used, where the Boltzmann equation can be written in the center of mass coordinate system as following:

$$\begin{aligned} \frac{Df_s}{Dt} + \mathbf{C}_s \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \left(\frac{e_s}{m_s} \mathbf{E} + \frac{e_s}{m_s} \mathbf{C}_0 \times \mathbf{H} - \frac{D\mathbf{C}_0}{Dt} \right) \cdot \frac{\partial f_s}{\partial \mathbf{C}_s} \\ + \frac{e_s}{m_s} (\mathbf{C}_s \times \mathbf{H}) \cdot \frac{\partial f_s}{\partial \mathbf{C}_s} - \frac{\partial f_s}{\partial \mathbf{C}_s} \cdot \mathbf{C}_s \cdot \frac{\partial \mathbf{C}_0}{\partial \mathbf{r}} = \sum_i J_{si}(f_s f_i), \end{aligned} \quad (1)$$

with

$$J_{si}(f_s f_i) = \iint (f'_s f'_i - f_s f_i) g_{si} \sigma_{si} d\Omega d\mathbf{v}_i, \quad (2)$$

where \mathbf{C}_s represents the peculiar velocity given by

$$\mathbf{C}_s = \mathbf{v}_s - \mathbf{C}_0, \quad (3)$$

and \mathbf{C}_0 represents the mean velocity of the mixture defined as

$$\rho \mathbf{C}_0 = \sum_s \rho_s \mathbf{u}_s, \quad (4)$$

$$\rho = \sum_s \rho_s = \sum_s n_s m_s \quad (\text{mass density}), \quad (4a)$$

In order to include the internal electronic degrees of freedom, we write

$$f_s = \sum_p f_{sp}(\mathbf{v}_s, \mathbf{r}, t, p), \quad (5)$$

where p denotes the quantum state, and the mean velocity of the constituent gas of "s" kind is given by

$$u_s = \frac{1}{n_s} \sum_p \int \mathbf{v}_s f_{sp} d\mathbf{v}_s, \quad (6)$$

where n_s , the number density, is defined by $n_s = \sum_p \int f_{sp} d\mathbf{v}_s$, E and H denote the strength of electric field and magnetic field respectively, e_s the electric charge, m_s the mass of the constituent gas of s kind, \mathbf{g}_{sj} the relative velocity between constituent gas s and j , as $\mathbf{g}_{sj} = \mathbf{v}_s - \mathbf{v}_j$, the $d\Omega$ denotes the solid angle and σ_{sj} denotes the collision cross section, (the measure of the interaction between particles of s and j kinds).

By multiplying a molecular property $\varphi_s(\mathbf{v}_s)$ and integrating over all the velocity space \mathbf{v}_s , and with the definition

$$\bar{\varphi}_s(\mathbf{r}, t) = \frac{1}{n_s} \sum_p \int f_{sp} \varphi_s(\mathbf{v}_s) d\mathbf{v}_s, \quad (7)$$

the equation of transport for individual component gas becomes

$$\begin{aligned} & \frac{D}{Dt} (n_s \bar{\varphi}_s) + n_s \bar{\varphi}_s \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{C}_0 + \frac{\partial}{\partial \mathbf{r}} \cdot (n_s \bar{\varphi}_s \mathbf{C}_s) \\ & - n_s \left\{ \frac{D\bar{\varphi}_s}{Dt} + \mathbf{C}_s \cdot \frac{\partial \bar{\varphi}_s}{\partial \mathbf{r}} + \left(\frac{e_s}{m_s} E + \frac{e_s}{m_s} \mathbf{C}_0 \times H - \frac{D\mathbf{C}_0}{Dt} \right) \cdot \frac{\partial \bar{\varphi}_s}{\partial \mathbf{C}_s} \right. \\ & \left. + \frac{e_s}{m_s} (\mathbf{C}_s \times H) \cdot \frac{\partial \bar{\varphi}_s}{\partial \mathbf{C}_s} - \frac{\partial \bar{\varphi}_s}{\partial \mathbf{C}_s} \mathbf{C}_s : \frac{\partial}{\partial \mathbf{r}} \mathbf{C}_0 \right\} = \Delta (n_s \bar{\varphi}_s), \end{aligned} \quad (8)$$

with

$$\begin{aligned}\Delta(n_s \bar{\varphi}_s) &= \sum_i J_{si} (\varphi_s) \\ &= \sum_i \iiint \varphi_s (f'_s f_i - f_s f'_i) g_{si} \sigma_{si} d\Omega d\mathbf{v}_s d\mathbf{v}_i, \quad (9)\end{aligned}$$

The conservation equations immediately follow by taking $\varphi_s = 1$, $m_s \mathbb{C}_s$, and $\frac{1}{2} m_s C_s^2 + \epsilon_{sp}$ where ϵ_{sp} represents the internal energy of s specie at quantum state p , thus

$$\frac{Dn_s}{Dt} + n_s \frac{\partial}{\partial r} \cdot \mathbb{C}_o + \frac{\partial}{\partial r} \cdot (n_s \bar{\mathbb{C}}_s) = \Delta n_s, \quad (\text{eq. of continuity}), \quad (10)$$

$$\begin{aligned}& \rho_s \frac{D\bar{\mathbb{C}}_s}{Dt} + \rho_s \bar{\mathbb{C}}_s \cdot \frac{\partial}{\partial r} \mathbb{C}_o + \frac{\partial}{\partial r} \cdot (\rho_s \bar{\mathbb{C}}_s \mathbb{C}_s) - \bar{\mathbb{C}}_s \frac{\partial}{\partial r} \cdot (\rho_s \bar{\mathbb{C}}_s) \\ & - \rho_s \left(\frac{e_s}{m_s} |E| + \frac{e_s}{m_s} \mathbb{C}_o \times |H| - \frac{D\mathbb{C}_o}{Dt} \right) - n_s e_s (\bar{\mathbb{C}}_s \times H) \\ & = \rho_s \Delta \bar{\mathbb{C}}_s, \quad (\text{eq. of momentum conservation}), \quad (11)\end{aligned}$$

$$\begin{aligned}& \frac{D}{Dt} \left(\frac{1}{2} \rho_s \bar{C}_s^2 \right) + \frac{1}{2} \rho_s \bar{C}_s^2 \frac{\partial}{\partial r} \cdot \mathbb{C}_o + \frac{\partial}{\partial r} \cdot \left(\frac{1}{2} \rho_s \bar{\mathbb{C}}_s^2 \mathbb{C}_s \right) \\ & - \rho_s \bar{\mathbb{C}}_s \cdot \left(\frac{e_s}{m_s} |E| + \frac{e_s}{m_s} \mathbb{C}_o \times |H| - \frac{D\mathbb{C}_o}{Dt} \right) + \rho_s \bar{\mathbb{C}}_s \mathbb{C}_s : \frac{\partial}{\partial r} \mathbb{C}_o \\ & = \Delta \left(\frac{1}{2} \rho_s \bar{C}_s^2 \right), \quad (\text{eq. of energy conservation}), \quad (12)\end{aligned}$$

Defines:

a) Pressure tensor of component gas

$$P_s = \rho_s \overline{C_s C_s} = \sum_p \int m_s C_s C_s f_{sp} dC_s, \quad (13)$$

b) Heat flux of component gas

$$q_s = n_s \overline{E_s C_s} = \left(\frac{1}{2} \rho_s C_s^2 + e_{sp} \right) \overline{C_s} = \sum_p \int \left(\frac{1}{2} \rho_s C_s^2 + e_{sp} \right) f_{sp} C_s dC_s. \quad (14)$$

c) Conduction current of component gas

$$J_s = n_s e_s \overline{C_s}, \quad (15a)$$

$$\text{and } J_{so} = n_s e_s \overline{C_o}, \quad (15b)$$

Therefore, Eq. (11) and (12) become:

$$\begin{aligned} \rho_s \frac{D\overline{C_s}}{Dt} + \rho_s \overline{C_s} \cdot \frac{\partial}{\partial r} C_o + \frac{\partial}{\partial r} \cdot P_s - \overline{C_s} \frac{\partial}{\partial r} \cdot (\rho_s \overline{C_s}) - n_s e_s E \\ - J_{so} \times H + \rho_s \frac{DC_o}{Dt} - J_s \times H = \rho_s \Delta \overline{C_s}, \end{aligned} \quad (16a)$$

$$\begin{aligned} \frac{D}{Dt} \left(\frac{1}{2} \rho_s \overline{C_s^2} \right) + \frac{1}{2} \rho_s \overline{C_s^2} \left(\frac{\partial}{\partial r} \cdot C_o \right) + \frac{\partial}{\partial r} \cdot q_s - \overline{C_s} \cdot n_s e_s E \\ - (J_{so} \times H) \cdot \overline{C_s} + \rho_s \overline{C_s} \cdot \frac{DC_o}{Dt} + P_s : \frac{\partial}{\partial r} C_o = \Delta \left(\frac{1}{2} \rho_s \overline{C_s^2} \right), \end{aligned} \quad (16b)$$

III. METHOD OF APPROXIMATION TO THE BOLTZMANN EQUATION.

The Chapman-Enskog method of solution is to employ the subdivision of the Boltzmann equation together with the expansion of the solution. Thus, first let us expand the solution and write

$$f_{sp} = f_{sp}^{(0)} + f_{sp}^{(1)} + f_{sp}^{(2)} + \dots, \quad (17)$$

where $f_{sp}^{(0)}$ denotes the equilibrium distribution function and $f_{sp}^{(r)}$ ($r > 1$) are the corrections for the non-LTE. In order to separate the translational and internal degrees of freedom, follow the Born Oppenheimer's approximation; thus

$$f_{sp} = f_s^{tr} f_s^{in}(p), \quad (18)$$

where $f_s^{tr}(w_i, r, t)$ represents the distribution related to the translational degrees of freedom and $f_s^{in}(p, r, t) = n_s(p, r, t)$ represents that of the internal degrees of freedom. Then the expansion of Eq. (18) becomes

$$\begin{aligned} f_{sp} &= f_s^{tr} f_s^{in}(p) = (f_s^{tr(0)} + f_s^{tr(1)} + \dots) (f_s^{in(0)}(p) + f_s^{in(1)}(p) + \dots) \\ &= f_s^{tr(0)} f_s^{in(0)}(p) + f_s^{tr(0)} f_s^{in(0)}(p) \psi_s^{tr} + f_s^{tr(0)} f_s^{in(0)}(p) \psi_s^{in}(p) \dots, \end{aligned} \quad (19)$$

where we write

$$f_s^{tr(1)} = f_s^{tr(0)} \psi_s^{tr}, \text{ and } f_s^{in(1)}(p) = f_s^{in(0)}(p) \psi_s^{in}(p), \quad (20)$$

with ψ_s^{tr} denoting the usual correction proportional to the spatial gradients of the macroscopic properties and $\psi_s^{in}(p)$ denoting the correction for population density under non-LTE.

Using the equilibrium distribution function given by Nakagawa [6] and Wu [1], we have

$$f_{sp}^{(0)} = F_s^{in(0)}(p) F_s \left[1 + \left(\frac{m_s C_s^2}{2kT_o} - \frac{3}{2} \right) \alpha_s + \frac{m_s}{kT_o} C_s \cdot \beta_s + \dots \right], \quad (21)$$

with $F_s^{in(0)}(p)$ and F_s denoting Boltzmann and Maxwell distribution functions referred to the center of mass system of the mixture respectively; i.e.

$$F_s^{in(0)}(p) = n_s \frac{g_{sp} \exp(-\epsilon_{sp}/kT_o)}{\sum_p g_{sp} \exp(-\epsilon_{sp}/kT_c)}, \quad (22)$$

and

$$F_s = \left(\frac{m_s}{2\pi kT_o} \right)^{3/2} \exp\left(-\frac{m_s C_s^2}{2kT_o}\right), \quad (23)$$

where T_o represents the temperature of the mixture,

$$T_o = \frac{\sum_s n_s T_s}{n}, \quad (24)$$

and

$$\alpha_s = \frac{T_s - T_o}{T_s}, \quad (25)$$

$$\beta_s = u_s - C_o, \quad (26)$$

Similar to the BGK modeling [7], we shall only retain the terms α_s and β_s up to their first order. The uniqueness of this expansion can be secured by requiring the conservation of mass, momentum and energy with respect to \mathbb{C}_0 and T_0 system, which are

$$\left. \begin{aligned} \sum_s \sum_p \int m_s f_s^{(r)} d\mathbb{C}_s &= 0, \\ \sum_s \sum_p \int f_s^{(r)} m_s \mathbb{C}_s d\mathbb{C}_s &= 0, \\ \sum_s \sum_p \int f_s^{(r)} \left(\frac{1}{2} m_s \mathbb{C}_s^2 + \epsilon_{sp} \right) d\mathbb{C}_s &= 0. \end{aligned} \right\} (r \geq 1). \quad (27)$$

Now returning to Eq. (1) we may seek the successive approximation of the solution f_s by introducing appropriate successive subdivision of the equation. Following the work of Chapman and Cowling [2], rewriting Eq. (1) in an operator form,

$$D_s - J_{sj} = 0, \quad (28)$$

and its subdivision

$$D_s^{(r)} - J_{sj}^{(r)el} - J_{sj}^{(r)non-el} = 0, \quad (29)$$

where

$$D_s = \sum_r D_s^{(r)} \text{ and } J_{sj} = \sum_r (J_{sj}^{(r)el} + J_{sj}^{(r)non-el}), \quad (30)$$

The Enskog's method requires the subdivision of first order approximation to be the vanishing collision integral, and the next order approximation to be balance the spatial dependence with the first order collision integral. With the mentioned scheme of approximation, the governing equation for successive order can be obtained in the following manner:

To the first order of present approximation;

$$\frac{e_s}{m_s} (\mathbf{C}_s \times \mathbf{H}) \cdot \frac{\partial F_s}{\partial \mathbf{C}_s} = - \sum_i J_{si}^{(0)el} (F_s F_i) + \sum_i J_{si}^{(0)non-el} (F_s^{in(0)}(p) F_i^{in(0)}(q)) = 0, \quad (31)$$

As we should note, the expression of $F_s^{in(0)}(p)$ and F_s given by Eqs. (22) and (23) do indeed satisfy the condition of Eq. (31). Therefore, they are the first order solution of the Boltzmann equation (i.e., equilibrium solution).

To the second order of the present approximation we find;

$$\begin{aligned} \sum_p \left[\frac{D_o f^{(0)}_{sp}}{Dt} + \mathbf{C}_s \cdot \frac{\partial f^{(0)}_{sp}}{\partial \mathbf{r}} + \left(\frac{e_s}{m_s} \mathbf{E} - \frac{D_o \mathbf{C}_o}{Dt} \right) \cdot \frac{\partial f^{(0)}_{sp}}{\partial \mathbf{C}_s} \right. \\ \left. + \frac{e_s}{m_s} (\mathbf{C}_o \times \mathbf{H}) \cdot \frac{\partial f^{(0)}_{sp}}{\partial \mathbf{C}_s} - \frac{\partial f^{(0)}_{sp}}{\partial \mathbf{C}_s} \mathbf{C}_s \cdot \frac{\partial}{\partial \mathbf{r}} \mathbf{C}_o \right] \\ = - \frac{e_s}{m_s} F_s (\mathbf{C}_s \times \mathbf{H}) \cdot \frac{\partial}{\partial \mathbf{C}_s} (\Psi_s^{tr} + \delta_s) - \sum_i J_{si}^{(1)el} [F_s F_i (\delta_s + \delta_i)] \\ - \sum_i J_{si}^{(2)el} [F_s F_i (\Psi_s^{tr} + \Psi_i^{tr})], \end{aligned} \quad (32)$$

where δ_s denotes the correction terms involved α_s and β_s . (We should note the term $J_{sj}^{(2) \text{ non-el}}$ has been dropped, because it is few orders of magnitude smaller than the term $J_{sj}^{(2) \text{ el}}$ (i.e., $\sigma_{sj}^{\text{non-el}} / \sigma_{sj}^{\text{el}} \ll 1$).

Substituting the expression of $f_{sp}^{(0)}$ from Eq. (21) into Eq. (32), we find,

$$\begin{aligned}
 J_{sj}^{(2) \text{ el}} [F_s F_j (\psi_s^{\text{tr}} + \psi_j^{\text{tr}})] = F_s \left\{ \frac{D_o \ln n_s}{Dt} + \mathcal{C}_s \cdot \frac{\partial \ln n_s}{\partial r} \right. \\
 + \left[\left(\frac{m_s C_s^2}{2kT_o} - \frac{3}{2} \right) - \left(\frac{m_s C_s^2}{2kT_o} \alpha_s + \frac{m_s}{kT_o} \mathcal{C}_s \cdot \beta_s \right) + \frac{e_{sp} - \bar{e}_s}{kT_o} \right] \cdot \frac{D_o \ln T_o}{Dt} \\
 + \mathcal{C}_s \cdot \left[\left(\frac{m_s C_s^2}{2kT_o} - \frac{3}{2} \right) - \left(\frac{m_s C_s^2}{2kT_o} \alpha_s + \frac{m_s}{kT_o} \mathcal{C}_s \cdot \beta_s \right) + \frac{e_{sp} - \bar{e}_s}{kT_o} \right] \frac{\partial \ln T_o}{\partial r} \\
 + \left(\frac{m_s C_s^2}{2kT_o} - \frac{3}{2} \right) \frac{D_o \alpha_s}{Dt} + \frac{m_s}{kT_o} \mathcal{C}_s \cdot \frac{D_o \beta_s}{Dt} \\
 + \mathcal{C}_s \cdot \left(\frac{m_s C_s^2}{2kT_o} - \frac{3}{2} \right) \frac{\partial \alpha_s}{\partial r} + \frac{m_s}{kT_o} \mathcal{C}_s \mathcal{C}_s : \frac{\partial \beta_s}{\partial r} \\
 - \left(\frac{e_s}{m_s} E - \frac{D_o \mathcal{C}_o}{Dt} \right) \cdot \frac{m_s}{kT_o} \mathcal{C}_s + \left(\frac{e_s}{m_s} E - \frac{D_o \mathcal{C}_o}{Dt} \right) \cdot \left(\frac{m_s}{kT_o} \alpha_s + \frac{m_s \beta_s}{kT_o} \right) \\
 + \frac{m_s}{kT_o} (1 - \alpha_s) \mathcal{C}_s \mathcal{C}_s : \frac{\partial}{\partial r} \mathcal{C}_o - \frac{m_s}{kT_o} \beta_s \mathcal{C}_s : \frac{\partial}{\partial r} \mathcal{C}_o \\
 \left. - \frac{e_s}{m_s} (\mathcal{C}_o \times H) \cdot \frac{m_s \mathcal{C}_s}{kT_o} + \frac{e_s}{m_s} (\mathcal{C}_o + \mathcal{C}_s) \times H \cdot \left(\frac{m_s \mathcal{C}_s}{kT_o} \alpha_s + \frac{m_s \beta_s}{kT_o} \right) \right\},
 \end{aligned} \tag{33}$$

with

$$\frac{D_o}{Dt} = \frac{\partial}{\partial t} + \mathcal{C}_o \cdot \frac{\partial}{\partial r}, \tag{33a}$$

and

$$\bar{e}_s = \frac{\sum_p e_{sp} g_{sp} \exp[-e_{sp}/kT_o]}{\sum_p g_{sp} \exp[-e_{sp}/kT_o]} \tag{33b}$$

In order to eliminate the time derivatives (i.e., $\frac{D_o \ln T_o}{Dt}$, $\frac{D \mathcal{C}_o}{Dt}$... etc.) from previous expression, the appropriate conservation equations to this first order of approximation are needed, and are;

for number density,

$$\frac{D_o n_s}{Dt} + n_s \frac{\partial}{\partial r} \cdot \mathcal{C}_s + \frac{\partial}{\partial r} \cdot (n_s \beta_s) = 0, \quad (34)$$

for momentum,

$$\begin{aligned} \rho_s \frac{D_o \beta_s}{Dt} + \frac{\partial}{\partial r} p_s + \rho_s \frac{D_o \mathcal{C}_o}{Dt} + \rho_s \beta_s \frac{\partial}{\partial r} \cdot \mathcal{C}_o - \rho_s \left[\frac{e_s}{m_s} |E + \frac{e_s}{m_s} (\mathcal{C}_o + \beta_s) \times H| \right] \\ = \sum_i \frac{m_s (\beta_{si} - \beta_s)}{\tau_{el}^{m_{sj}}}, \end{aligned} \quad (35)$$

for energy,

$$\begin{aligned} \frac{3}{2} n_s k T_o \frac{D_o \alpha_s}{Dt} + n_s \left[\frac{3}{2} k (1 + \alpha_s) + n_s c_{s, \text{int}} \right] \frac{D_o T_o}{Dt} \\ + n_s \left(\frac{5}{2} k + c_{s, \text{int}} \right) \beta_s \cdot \frac{\partial T_o}{\partial r} + p_s \frac{\partial}{\partial r} \cdot \mathcal{C}_o + \rho_s \beta_s \cdot \frac{D_o \mathcal{C}_o}{Dt} \\ + k T_o \frac{\partial}{\partial r} \cdot (n_s \beta_s) - \rho_s \beta_s \left(\frac{e_s}{m_s} \mathcal{C}_o \times H \right) = \sum_i \frac{\alpha_{si} - \alpha_s}{\tau_{E_{sj}}}, \end{aligned} \quad (36)$$

For the total gas

$$\rho \frac{D_o \mathcal{C}_o}{Dt} + \mathbf{J}_{so} \times \mathbf{H} = - \frac{\partial}{\partial r} p \quad (37)$$

and

$$\begin{aligned} & \left(\frac{3}{2} nk + \sum_s n_s c_{s, \text{int}} \right) \frac{D_o T_o}{Dt} + \frac{3}{2} k T_o \frac{\partial}{\partial r} \cdot \left(\sum_s n_s \beta_s \right) \\ & + \sum_s \left[\left(\frac{5}{2} k + c_{s, \text{int}} \right) n_s \beta_s \cdot \frac{\partial T_o}{\partial r} \right] + p \frac{\partial}{\partial r} \cdot \mathcal{C}_o - \sum_s \left[\rho_s \beta_s \left(\frac{e_s}{m_s} E + \frac{e_s}{m_s} \mathcal{C}_o \times \mathbf{H} \right) \right] = 0 \end{aligned} \quad (38)$$

with

$$p_s = n_s (1 + \alpha_s) k T_o, \quad (39)$$

$$p = nk T_o = \sum_s p_s, \quad (40)$$

where $c_{s, \text{int}}$ is the specific heat due to internal (electronic) degrees of freedom

and is given by

$$c_{s, \text{int}} = \frac{\overline{\epsilon_s^2} - (\overline{\epsilon_s})^2}{k T_o}, \quad (41)$$

with

$$\overline{\epsilon_s^2} = \frac{\sum_p \epsilon_{sp}^2 g_{sp} \exp[-\epsilon_{sp}/k T_o]}{\sum_p g_{sp} \exp[-\epsilon_{sp}/k T_o]} \quad (42)$$

The symbols $\tau_{m_{sj}}^{el}$ and $\tau_{E_{sj}}^{el}$ represent the relaxation coefficients for momentum and energy respectively.

Substituting Eqs. (34) through (38) into Eq. (33) and retaining only those terms up to the first order of α_s and β_s , we find,

$$\begin{aligned}
 (2) \quad J_{sj}^{el} [F_s F_i (\Psi_s^{tr} \Psi_i^{tr})] = F_s \left\{ \left[\left(\frac{m_s C_s^2}{2kT_o} - \frac{5}{2} \right) - \frac{m_s C_s^2}{2kT_o} \alpha_s + \frac{e_{sp} - \bar{e}_s}{kT_o} \right] C_s \cdot \frac{\partial \ln T_o}{\partial r} \right. \\
 + \frac{n}{n_s} \alpha_s C_s \cdot \left[\left(\frac{\rho}{\rho_s} - \frac{n}{n_s} \right) \frac{\partial \ln \rho}{\partial r} - \frac{\partial}{\partial r} \left(\frac{n}{n_s} \right) \right] \\
 + \left(\frac{m_s C_s^2}{2kT_o} - \frac{5}{2} \right) C_s \cdot \frac{\partial \alpha}{\partial r} + (1 - \alpha_s) \frac{m_s}{kT_o} C_s^o C_s : \frac{\partial}{\partial r} C_o + \frac{m_s}{kT_o} C_s^o C_s : \frac{\partial \beta_s}{\partial r} \\
 + \frac{2}{3} \left[\frac{2}{3} \frac{C_s}{k} \text{int} \left(\frac{m_s C_s^2}{2kT_o} - \frac{3}{2} \right) + \frac{m_s C_s^2}{2kT_o} \alpha_s - \frac{e_{sp} - \bar{e}_s}{kT_o} \right] \left(\frac{\partial}{\partial r} \cdot C_o \right) \\
 + \frac{2}{3} \left(\frac{m_s C_s^2}{2kT_o} - \frac{3}{2} \right) \frac{n e_s C_s}{n_s kT_o} \cdot E + \left[\frac{2}{3} \left(\frac{m_s C_s^2}{2kT_o} - \frac{3}{2} \right) + 1 \right] \frac{\beta_s}{n_s kT_o} \cdot (J_{so} \times H) \\
 \left. + \frac{2\beta_s + \alpha_s C_o}{n_s kT_o} \cdot (n_s e_s C_s \times H) \right\}, \quad (43)
 \end{aligned}$$

The expression of Eq. (43) leads to the functional form of Ψ_s . Since Ψ_s , like $f_s^{(1)}$ itself, is a scalar, therefore, Ψ_s must be a scalar.

$$\Psi_s^{tr} = -/A_s \cdot \frac{\partial \ln T_o}{\partial r} - /B_s : \frac{\partial}{\partial r} C_o - G_s \cdot d_s - K_s \frac{\partial}{\partial r} \cdot C_o - /D_s \cdot \frac{\partial \alpha_s}{\partial r}$$

$$- \mathcal{H}_s : \frac{\partial}{\partial r} [\beta_s - /L_s \cdot /E - M_s [\beta_s \cdot (J_{so} \times H)]]$$

$$- /M /L_s \cdot (C_s \times H), \quad (44)$$

$$\text{with } \mathcal{H}_s = \left(\frac{\rho}{\rho_s} - \frac{n_s}{n} \right) \frac{\partial}{\partial r} p - \frac{\partial}{\partial r} \left(\frac{n_s}{n} \right) \quad (44a)$$

The coefficients $/A_s$, G_s , D_s , $/L_s$ and $/M /L_s$ are vectors, $/B_s$ and \mathcal{H}_s are tensors, and K_s and M_s are scalars; they can be expressed in the form

$$/A_s = \left(\frac{2kT_o}{m_s} \right)^{1/2} \left[(1 - \alpha_s) \mathcal{C}_{sm,n}^{\Sigma} a_{mn} S_{3/2}^m (\mathcal{C}_s^2) Y^n (\epsilon_{sp}) \right.$$

$$\left. - \alpha_s \mathcal{C}_{sm,n}^{\Sigma} a'_{mn} S_{3/2}^m (\mathcal{C}_s^2) Y^n (\epsilon_{sp}) \right],$$

$$/B_s = (1 - \alpha_s) \mathcal{C}_s^0 \mathcal{C}_{sm,n}^{\Sigma} b_{mn} S_{5/2}^m (\mathcal{C}_s^2) Y^n (\epsilon_{sp}),$$

$$/D_s = \left(\frac{2kT_o}{m_s} \right)^{1/2} \mathcal{C}_s^{\Sigma} d_{mn} S_{3/2}^m (\mathcal{C}_s^2) Y^n (\epsilon_{sp})$$

$$G_s = \left(\frac{2kT_o}{m_s} \right)^{1/2} \alpha_s \mathcal{C}_s^{\Sigma} g_{mn} S_{3/2}^m (\mathcal{C}_s^2) Y^n (\epsilon_{sp}),$$

$$\mathcal{H}_s = \mathcal{C}_s^0 \mathcal{C}_{sm,n}^{\Sigma} h_{mn} S_{5/2}^m (\mathcal{C}_s^2) Y^n (\epsilon_{sp}),$$

(45)

$$\begin{aligned}
K_s &= \frac{c_s}{k} \int \sum_{mn} k_{mn} S_{1/2}^m (\mathcal{C}_s^2) Y^n(\epsilon_{sp}) + \alpha_s \sum_{mn} k'_{mn} \left[S_{1/2}^m (\mathcal{C}_s^2) - \frac{3}{2} \right] Y^n(\epsilon_{sp}), \\
IL_s &= \frac{2 e_s}{3 k T_o} \left(\frac{2 k T_o}{m_s} \right)^{1/2} \sum_{m,n} L_{mn} \left[S_{3/2}^m (\mathcal{C}_s^2) + 1 \right] Y^n(\epsilon_{sp}), \\
M_s &= \frac{1}{n_s k T_o} \sum_{mn} M_{mn} \left[\frac{2}{3} S_{3/2}^m (\mathcal{C}_s^2) + \frac{5}{3} \right] Y^n(\epsilon_{sp}), \\
/MI_s &= \frac{e_s (2\beta_s + \alpha_s \mathcal{C}_o)}{k T_o} \sum_{m,n} M_{mn} S_{1/2}^m (\mathcal{C}_s^2) Y^n(\epsilon_{sp}).
\end{aligned}$$

$S_n^m(\mathcal{C}_s^2)$ is a sonine polynomial, used by Chapman and Cowling [2]

and Burnett [8]. $Y^n(\epsilon_{sp})$ is the n-th order polynomial used by Wang-Chang and Uhlenbeck [9], i.e.

$$\left. \begin{aligned}
Y^{(0)}(\epsilon_{sp}) &= 1 \\
Y^{(1)}(\epsilon_{sp}) &= \frac{\bar{\epsilon}_s - \epsilon_{sp}}{k T_o}
\end{aligned} \right\} \quad (46)$$

while \mathcal{C}_s is the dimensionless velocity

$$\mathcal{C}_s = \left(\frac{m_s}{2 k T_o} \right)^{1/2} \mathcal{C}_s. \quad (47)$$

The requirements Eq. (27), imposed on the solution lead to the following conditions,

$$\sum_s \sum_p \int F_s F_s^{\text{in}(0)}(p) m_s \mathbb{C}_s \begin{bmatrix} A_s \\ D_s \\ G_s \\ L_s \\ M_s \end{bmatrix} d\mathbb{C}_s = 0, \quad (48a)$$

$$\sum_s \sum_p \int F_s F_s^{\text{in}(0)}(p) \begin{bmatrix} B_s \\ \mathcal{H}_s \\ K_s \\ M_s \end{bmatrix} d\mathbb{C}_s = 0, \quad (48b)$$

$$\sum_s \sum_p \int F_s F_s^{\text{in}(0)}(p) \left(\frac{1}{2} m_s C_s^2 + \epsilon_{sp} \right) \begin{bmatrix} B_s \\ \mathcal{H}_s \\ K_s \\ M_s \end{bmatrix} d\mathbb{C}_s = 0, \quad (48c)$$

IV. OHM'S LAW FOR N-COMPONENT GAS MIXTURE UNDER NON-LTE IN THE FIRST ORDER APPROXIMATION

By taking $\varphi_s = e_s C_s$ in the Eq. (8) and together with the first order approximation of the distribution function of Eq. (21) leads to a generalized Ohm's law of first order of approximation, describing the condition of current in an N-component gas mixture in the presence of electric and magnetic field at the condition of non-LTE, we find

$$\begin{aligned} \frac{D}{Dt} J_s^{(0)} + J_s^{(0)} \frac{\partial}{\partial r} \cdot C_o + J_s^{(0)} \cdot \frac{\partial}{\partial r} C_o + \frac{\partial}{\partial r} \left[\frac{\rho_{es}}{\rho_s} p_s \right] - \frac{\rho_{es}}{\rho} \frac{\partial}{\partial r} p \\ - \frac{\rho_{es}^2}{\rho_s} (E + C_o \times H) - \frac{\rho_{es}}{\rho_s} J_s^{(0)} \times H = \frac{\rho_{es}}{\rho_s} \sum_i \frac{l\beta_i - l\beta_s}{\tau_{el} m_{sj}}, \end{aligned} \quad (49)$$

where

$$J_s^{(0)} = e_s \int C_s f_s^{(0)} dC_s = n_s e_s \beta_s, \quad (50)$$

is the consequence of Eq. (15a)

$$\rho_{es} = n_s e_s \quad (\text{charge density}), \quad (51)$$

and

$$(\tau_{m_{sj}}^{el})^{-1} = \frac{1}{3} \sqrt{\frac{2}{\pi}} n_s n_i \frac{\mu_{sj}^{7/2}}{(kT_o)^{5/2}} \int_0^\infty g_{sj}^4 e^{-\frac{\mu_{sj}}{kT_o}} g_{sj}^2 g_{sj}^M dg_{sj}, \quad (52)$$

with $\mu_{si} = \frac{m_s m_i}{m_s + m_i}$, the reduced mass and σ_{si}^M the cross section for momentum transfer. As we note, the expression of Eq. (49) is a form of generalized Ohm's law for the constituent gas in N-component gas mixture. For the total gas, we just take the summation of Eq. (49) over all the components, thus

$$\begin{aligned} \frac{\partial}{\partial t} \left(\sum_s \frac{\rho_{es}}{\rho_s} p_s \right) - \frac{\rho_e}{\rho} \frac{\partial}{\partial t} p - \sum_s \left(\frac{\rho_{es}^2}{\rho_s} \right) (\mathbf{E} + \mathbf{C}_0 \times \mathbf{H}) \\ - \sum_s \frac{\rho_{es}}{\rho_s} \mathbf{J}_s^{(0)} \times \mathbf{H} = 0, \end{aligned}$$

with $\rho_e = \sum_s \rho_{es}$ (total charge density) . (53)

This expression represents the generalized Ohm's law, to first order approximation, of a gas mixture under the condition of non-LTE. In other words, this gives the law of conduction of electric current in a non-LTE gas mixture. This form of Ohm's law can be used in problems of the motion of gas with anisotropic conductivity and where non-LTE conditions prevail, such as certain regions of the solar photosphere.

V. OHM'S LAW FOR N-COMPONENT GAS MIXTURE UNDER NON-LTE IN THE SECOND APPROXIMATION

To determine Ohm's law in the form given by Eq. (49) we omitted the correction term in the distribution function of Eq. (19). This is due to the assumption that the gradients of macroscopic quantities are negligible small, but this is not always a valid assumption. In particular, when the gradients become appreciable, i.e., the situation in which deviations from equilibrium become significant, such effects have to be included in order to get a more realistic description. We shall go to the next order approximation by using the distribution function given by Eq. (17) and (20) together with expression of Eq. (44) for the correction function Ψ_s . Again, by putting $\varphi_s = e_s C_s$ in Eq. (8), we obtain the form of the generalized Ohm's law to the second order of approximation as follows:

$$\begin{aligned} & \frac{D}{Dt} (J_s) + J_s \frac{\partial}{\partial r} \cdot C_o + J_s \cdot \frac{\partial}{\partial r} C_o + \frac{\partial}{\partial r} \left(\frac{\rho_{es}}{\rho_s} p_s \right) \\ & - \frac{\rho_{es}^2}{\rho_s} (E + C_o \times H) + \rho_{es} \frac{D C_o}{Dt} - \frac{\rho_{es}}{\rho_s} J_s \times H = \frac{\rho_{es}}{\rho_s} \sum_i \frac{|\beta_i - \beta_s|}{\tau_{m_{si}}} \end{aligned} \quad (54)$$

with

$$J_s = \int e_s C_s f_s^{(0)} (1 + \Psi_s^{tr}) d C_s, \quad (55)$$

more specifically, we find

$$\mathbb{J}_s = n_{ss} e \beta_s + \sum_p \int e \mathbb{C}_{s sp} f_s^{(0)} \Psi_s^{\text{tr}} d\mathbb{C}_s, \quad (56)$$

and the second term on the right hand side of Eq. (56) can be calculated further, thus

$$\begin{aligned} \sum_p \int e \mathbb{C}_{s sp} f_s^{(0)} \Psi_s^{\text{tr}} d\mathbb{C}_s &= \sum_p e \int \mathbb{C}_{s sp} F_s^{\text{in}(0)}(p) \left[-A_s \cdot \frac{\partial \ln T_o}{\partial r} - B_s \cdot \frac{\partial}{\partial r} \mathbb{C}_o \right. \\ &\quad - G_s \cdot \frac{dl}{ds} - K_s \frac{\partial}{\partial r} \cdot \mathbb{C}_o - D_s \cdot \frac{\partial \alpha_s}{\partial r} \\ &\quad - H_s \cdot \frac{\partial}{\partial r} \beta_s - L_s \cdot |E| \\ &\quad - M_s (\beta_s \cdot (\mathbb{J}_{so} \times |H|)) \\ &\quad \left. - N_s \cdot (\mathbb{C}_s \times |H|) \right] d\mathbb{C}_s \end{aligned}$$

where only the first order terms be kept.

Together with Eq. (45) above integration can be performed thus we find

$$\sum_p \int e \mathbb{C}_{s sp} f_s^{(0)} \Psi_s^{\text{tr}} d\mathbb{C}_s = -A_1 \frac{\partial T_s}{\partial r} - A_2 \frac{dl}{ds} - A_3 |E| - A_4 (\mathbb{J}_{so} \times |H|) - A_5 \frac{\partial \alpha_s}{\partial r}. \quad (57)$$

in which the terms of order of magnitude higher than ϵ have been left out, while the terms of spatial gradient and δ_s are classified as the order of magnitude of ϵ .

where

$$A_1 = \frac{n_s e}{\pi^{3/2}} \int \mathbb{C}_s \left(\frac{m_s}{2kT_o} \right) e^{-\mathbb{C}_s^2} \left[(1 - \alpha_s) \mathbb{E}_s \sum_{m,n} a_{mn} S_{3/2}^m (\mathbb{C}_s^2) Y^n(\epsilon_{sp}) - \alpha_s \mathbb{E}_s \sum_{m,n} a'_{mn} S_{3/2}^m (\mathbb{C}_s^2) Y^n(\epsilon_{sp}) \right] \frac{1}{T_o} d\mathbb{C}_s, \quad (58a)$$

$$A_2 = \frac{n_s e}{\pi^{3/2}} \int \mathbb{C}_s \left(\frac{m_s}{2kT_o} \right) e^{-\mathbb{C}_s^2} \alpha_s \mathbb{E}_s \sum_{m,n} g_{mn} S_{3/2}^m (\mathbb{C}_s^2) Y^n(\epsilon_{sp}) d\mathbb{C}_s, \quad (58b)$$

$$A_3 = \frac{n_s e}{\pi^{3/2}} \int \mathbb{C}_s \left(\frac{m_s}{2kT_o} \right) e^{-\mathbb{C}_s^2} \mathbb{E}_s \frac{2e_s}{3kT_o} \sum_{m,n} L_{mn} \left[S_{3/2}^m (\mathbb{C}_s^2) + 1 \right] Y^n(\epsilon_{sp}) d\mathbb{C}_s \quad (58c)$$

$$A_4 = \frac{n_s e \mathbb{E}_s \beta_s}{\pi^{3/2}} \int \mathbb{C}_s \left(\frac{m_s}{2kT_o} \right)^{3/2} e^{-\mathbb{C}_s^2} \left[\frac{1}{n_s kT_o} \sum_{m,n} M_{mn} \left(\frac{2}{3} S_{3/2}^m (\mathbb{C}_s^2) + \frac{5}{3} \right) Y^n(\epsilon_{sp}) \right] d\mathbb{C}_s. \quad (58d)$$

$$A_5 = \frac{n_s e}{\pi^{3/2}} \int \mathbb{C}_s \left(\frac{m_s}{2kT_o} \right) e^{-\mathbb{C}_s^2} \mathbb{E}_s \sum_{m,n} d_{mn} S_{3/2}^m (\mathbb{C}_s^2) Y^n(\epsilon_{sp}) d\mathbb{C}_s, \quad (58e)$$

Thus, the form of current density to the present order of approximation can be written as:

$$\mathbb{J}_s = n_s e \mathbb{E}_s - A_1 \frac{\partial T}{\partial r} - A_2 \frac{dI}{ds} - A_3 E - A_4 (\mathbb{J}_{so} \times \mathbb{H}) - A_5 \frac{\partial \alpha_s}{\partial r} \quad (59)$$

It is clear that the first term on the right hand side of Eq. (59) indicates the effect due to the difference of mean motion of each constituent gas, (i.e., macroscopic non-LTE in velocity), the second term represents the contribution from the temperature gradient (i.e., thermal conduction) which will become very important in calculations of shock structure and some solar phenomena, the third term represents the diffusion effects which include both ordinary diffusion (i.e., local gradient of concentration) and thermal diffusion, and the last two terms indicate the contributions from the electric and magnetic fields. We should note that the energy distribution function of each constituent gas has been modified according to the conditions of non-LTE. These effects are incorporated within the coefficients A_1 , A_2 , A_3 , A_4 , and A_5 , as can be observed from the expressions of Eq. (58). These coefficients are called transport coefficients, and will be determined by the coefficients of the expansions of Eq. (45), so-called "Kinetic coefficients", and the Boltzmann transport equation. This will be given in the following section.

VI. CALCULATION OF THE KINETIC COEFFICIENTS

In order to give a complete description for the generalized Ohm's law, the transport coefficients have to be given in the form of the appropriate kinetic coefficients. To determine these kinetic coefficients in terms of the expressions of the collision integrals, we shall employ the variational principle which was originated by Curtiss and Herschfelder [10] in this study.

1. General Statement of the Method

By substituting Eq. (44) into the L.H.S. of Eq. (43) and equating the corresponding coefficients of $\frac{\partial \ln T_0}{\partial r}$, $\frac{\partial C_0}{\partial r}$, dl_s etc. on both sides, we find that A_s , B_s , D_s , G_s , H_s , K_s , L_s , M_s and N_s are special solutions of the following equations.

$$F_s^{in(o)} \left[\left(\frac{m_s C_s^2}{2kT_0} - \frac{5}{2} \right) - \frac{m_s C_s^2}{2kT_0} \alpha_s + \frac{\epsilon_{sp} - \bar{\epsilon}_s}{kT_0} \right] C_s = n_s \sum_j n_j I_{sj} (A_s + A_j),$$

$$F_s^{in(o)} \left[\frac{n}{n_s} \alpha_s C_s (\delta_{sh} - \delta_{sk}) \right] = n_s \sum_j n_j I_{sj} (G_s^{(h)} + G_j^{(h)} + G_s^{(k)} + G_j^{(k)}),$$

$$F_s^{in(o)} \frac{m_s}{kT_0} (1 - \alpha_s) C_s^0 C_s = n_s \sum_j n_j I_{sj} (B_s + B_j),$$

$$\frac{2}{3} F_s^{in(o)} \left[\frac{2}{3} \frac{c_{sj}}{k} \int \left(\frac{m_s C_s^2}{2kT_0} - \frac{3}{2} \right) + \frac{m_s C_s^2}{2kT_0} \alpha_s - \frac{\epsilon_{sp} - \bar{\epsilon}_s}{kT_0} \right]$$

$$= n_s \sum_j n_j I_{sj} (K_s + K_j),$$

$$F_s^{in(o)} \frac{m_s}{kT_0} C_s^0 C_s = n_s \sum_j n_j I_{sj} (H_s + H_j),$$

> (60)

$$F_s^{in(\omega)} \left(\frac{m_s C_s^2}{2kT_0} - \frac{5}{2} \right) C_s = n_s \sum_j n_j I_{sj} (D_s + D_j),$$

$$\frac{2}{3} F_s^{in(\omega)} \left(\frac{m_s C_s^2}{2kT_0} - \frac{3}{2} \right) \frac{e_s C_s}{kT_0} = n_s \sum_j n_j I_{sj} (L_s + L_j),$$

$$F_s^{in(\omega)} \left[\frac{2}{3} \left(\frac{m_s C_s^2}{2kT_0} - \frac{3}{2} \right) + 1 \right] \frac{\beta_s}{n_s kT_0} = n_s \sum_j n_j I_{sj} (M_s \beta_s + M_j \beta_j),$$

$$F_s^{in(\omega)} \frac{e_s (2\beta_s + \alpha_s C_0)}{n_s kT_0} \cdot C_0 = n_s \sum_j n_j I_{sj} (M_s l_s \cdot C_s + M_j l_j \cdot C_j).$$

where the notation I_{sj} has its usual meaning as defined by Chapman and Cowling [2] and the δ_{sh} is the kronecker delta.

As we note, these sets of integral equations (Eq. (60)) given above, are quite similar in form; thus we can write one general form for a tensor Π_s , which includes all nine of these integral equations such as that

$$\mathcal{R}_s = n_s \sum_j n_j I_{sj} (\Pi_s + \Pi_j), \quad (61)$$

with

$$n_s \sum_j n_j I_{sj} (\Pi_s + \Pi_j) = 2\pi n_s \sum_{j,p} \int (\Pi_s' + \Pi_j' - \Pi_s - \Pi_j) F_j^{in(\omega)} F_j g_{sj} \sigma_{sj}^{el} d\mathbf{v}_j, \quad (62)$$

where σ_{sj}^{el} denotes the elastic collision cross section. The correspondence between the symbol \mathcal{R}_s and Π_s and their counterparts in Eq. (60) will be shown on Table I. The solution of the integral equation (61) is not unique, but this problem of uniqueness can be resolved by the solubility conditions posed on those functions A_s, B_s , etc., which can be also expressed in a general form of

$$\sum_s n_s \int (\Pi_s \cdot C_s) F_s dC_s = 0, \quad (63)$$

Table I

\mathcal{R}_s	\mathcal{T}_s
$F_s \left[\left(\frac{m_s C_s^2}{2kT_0} - \frac{5}{2} \right) - \frac{m_s C_s^2}{2kT_0} \alpha_s + \frac{e_{sp} - \bar{e}_s}{kT_0} \right] C_s$	\mathcal{A}_s
$F_s \left[\frac{n}{n_s} \alpha_s C_s (\delta_{sh} - \delta_{sk}) \right]$	\mathcal{G}_s
$F_s \frac{m_s}{kT_0} (1 - \alpha_s) C_s^{\circ} C_s$	\mathcal{B}_s
$\frac{2}{3} F_s \left[\frac{2}{3} \frac{c_{s, int}}{k} \left(\frac{m_s C_s^2}{2kT_0} - \frac{3}{2} \right) + \frac{m_s C_s^2}{2kT_0} \alpha_s - \frac{e_{sp} - \bar{e}_s}{kT_0} \right]$	\mathcal{K}_s
$F_s \frac{m_s}{kT_0} C_s^{\circ} C_s$	\mathcal{H}_s
$F_s \left(\frac{m_s C_s^2}{2kT_0} - \frac{5}{2} \right) C_s$	\mathcal{D}_s
$\frac{2}{3} F_s \left(\frac{m_s C_s^2}{2kT_0} - \frac{3}{2} \right) \frac{e_s C_s}{kT_0}$	\mathcal{L}_s
$F_s \left[\frac{2}{3} \left(\frac{m_s C_s^2}{2kT_0} - \frac{3}{2} \right) + 1 \right] \frac{\beta_s}{n_s kT_0}$	$\mathcal{M}_s \beta_s$
$F_s \frac{e_s (2\beta_s + \alpha_s C_0)}{n_s kT_0} \cdot C_0$	$\mathcal{M}_s \mathcal{I}_s \cdot C_s$

which, in fact, requires that the value of n_s , C_s , and T_s satisfy the conservation laws. Thus, the solubility conditions serve to specify exactly one of the solution of Eq. (60), i.e., they serve to remove an indeterminacy obtained. At present, we shall show how one can make use of the variational principles to obtain approximate solutions to the integral Eq. (60).

Let t_s be a set of functions which satisfy the equation

$$\int (t_s : \mathcal{R}_s) dv_s = - \sum_j n_s n_j [t_s : t_s + t_j]_{sj} \quad (64)$$

The bracket notation represents the integral

$$[G, H]_{sj} = \iint G (H - H') F_s F_j \sigma_{sj} dv_j dv_s, \quad (65)$$

where superscript prime indicates the quantity after collision. It is easily shown that this is identical with *

$$[G, H]_{sj} = -\frac{1}{2} \int (G - G') (H - H') F_s F_j \sigma_{sj} dv_j dv_s, \quad (66)$$

so that

$$[G, H]_{sj} = [G, H]_{js} = [H, G]_{sj} \quad (67)$$

From Eq. (61) and Eq. (64), we find

$$\sum_j n_s n_j [t_s : T_s + T_j]_{sj} = \sum_j n_s n_j [t_s : t_s + t_j]_{sj}, \quad (68)$$

where the t_s are the trial functions and T_s are the exact solutions to the integral Eq. (61). Now, if we sum Eq. (68) over s and make use of the symmetry property of Eq. (67). The result is

$$\{t, T\} = \{t, t\} \quad (69)$$

where the curly bracket represents

$$\{G, H\} = \sum_j n_s n_j [G_s + G_j, H_s + H_j]_{sj} \quad (70)$$

* Detail see Ref. [2] p. 67

It is clear that $\{ \}$ also possess the symmetry property

$$\{ G, H \} = \{ H, G \} \quad . \quad (71)$$

Since $\{ G, G \}$ is the sum of the integrals, the integrands of which are essentially positive,

$$\{ G, G \} \geq 0 \quad . \quad (72)$$

If G satisfies the solubility conditions, then the equality sign can apply only if G is identically zero.

Let us consider a particular curly bracket

$$\{ t - \mathbb{T}, t - \mathbb{T} \} > 0 \quad . \quad (73)$$

From the previous discussion, we note the equality sign can apply only if t is identically equal to \mathbb{T} . The curly bracket is a linear operator. Hence, from Eq. (73) we have,

$$\{ t, t \} - 2 \{ t, \mathbb{T} \} + \{ \mathbb{T}, \mathbb{T} \} \geq 0 \quad , \quad (74)$$

by using Eq. (69), we find

$$\{ t, t \} \leq \{ \mathbb{T}, \mathbb{T} \} \quad . \quad (75)$$

This is the statement of the variational method of obtaining approximations of the solution \mathbb{T}_s . Thus, the method of solution as follows; we begin by choosing a set of trial functions, t_s , which contain a number of arbitrary parameters. Then, if only those trial functions are considered which satisfy the solubility conditions (Eq. (63)), the equality sign in Eq. (75) applies only when \mathbb{T}_s and t_s are identical. Thus, the best approximation of the true solution of the integral Eq. (61) is obtained by maximizing $\{ t, t \}$ with respect to all the available parameters in the set of trial functions. That is, for the best approximation;

$$\delta \{ t, t \} = -2 \delta \sum_s \int (t_s : \mathcal{R}_s) d\mathbf{w}_s = 0, \quad (76)$$

This, along with Eq. (64), which restricts the choice of the trial function, forms the basis of the variational method of solution of the integral Eq. (61). For the present purpose of calculation of those kinetic coefficients, we consider the trial function which is known function as we have already chosen those functions of A_s, B_s, \dots etc., and which are expanded in a form of a double finite series of Eq. (45). Thus, we now take the trial function as a finite linear combination of this double finite series and its arguments are ξ_s^2 and ϵ_{sp} .

$$t_s = \tilde{W}_s \sum_{\substack{\xi=1 \\ n=0}}^{\xi-1} t_{smn}(\xi) S_n^m(\xi_s^2) Y^n(\epsilon_{sp}), \quad (77)$$

and these polynomials satisfy the orthogonality condition

$$\int x^n e^{-x} S_n^{m'}(x) S_n^{m''}(x) dx = \frac{(n+m)!}{m!} \delta_{mm'}, \quad (78)$$

with $\delta_{mm'}$ known as the kronecker delta. In which the values of the index n' and the meaning of the tensor \tilde{W}_s in Eq. (77) are shown in Table II.

Let us define

$$\mathcal{R}_{smn} = \sum_p \int (\mathcal{R}_s : \tilde{W}_s) S_n^m(\xi_s^2) Y^n(\epsilon_{sp}) d\mathbf{w}_s, \quad (79)$$

and

$$\Theta = \sum_{s, m, n} t_{smn} \mathcal{R}_{smn}, \quad (80)$$

The trial solution should satisfy the following equation

$$\int (t_s : \mathcal{R}_s) d\mathbf{w}_s = - \sum_j n_s n_j [t_s : t_s + t_j], \quad (81)$$

this is the constraint equation on the trial function, with the aid of Eqs. (77),

(79), and (80), we can rewrite Eq. (81) as such

Table II

T_s	n'	\tilde{W}_s	The series expansion coefficient t_{smn} are designated by
A_s	$3/2$	\mathcal{P}_s	a_{mn}, a'_{mn}
D_s	$3/2$	\mathcal{P}_s	d_{mn}
B_s	$5/2$	\mathcal{P}_s^0	b_{mn}
K_s	$1/2$	1	k_{mn}, k'_{mn}
\mathcal{A}_s	$5/2$	$\mathcal{P}_s^0 \mathcal{P}_s$	h_{mn}
G_s	$3/2$	$\alpha_s \mathcal{P}_s$	g_{mn}
L_s	$3/2$	$e_s \mathcal{P}_s$	L_{mn}
M_s	$3/2$	1	M_{mn}
MI_s	$1/2$	$e_s (2\beta_s + \alpha_s \mathcal{C}_0)$	MI_{mn}

$$\begin{aligned}
\omega_s = & \sum_{m,n=0}^{\xi-1} t_{smn} R_{smn} \\
& + \sum_j \sum_p \sum_{m=0}^{\xi-1} \sum_{n=0}^{\xi-1} n_s n_j \left\{ t_{smn} \left[\tilde{W}_s S_n^m Y^n; \tilde{W}_s S_n^{m''} Y^{n''} \right]_{s,j} \right. \\
& \left. + t_{jmn} \left[\tilde{W}_s S_n^m Y^n; \tilde{W}_j S_n^{m''} Y^{n''} \right]_{s,j} \right\} = 0. \quad (82)
\end{aligned}$$

According to the theory of variational principle, we had

$$\delta \Theta = 0. \quad (83)$$

Then, the problem is to find the extremum of Θ subject to the constraints of

Eq. (82). This extremum is determined by the method of lagrangian multipliers.

Let ι_s be the multiplier, thus

$$\frac{\partial \Theta}{\partial t_{smn}} + \sum_r \iota_r \left(\frac{\partial \omega_r}{\partial t_{smn}} \right) = 0. \quad (84)$$

Performing the indicated differentiations with the aid of Eqs. (80) and (82), we

obtain

$$\begin{aligned}
& [1 + \iota_s] R_{smn} \\
& + \sum_j \sum_{m=0}^{\xi-1} \sum_{n=0}^{\xi-1} \left\{ 2 \iota_s t_{smn} \left[\tilde{W}_s S_n^m (\mathcal{E}_s^2) Y^n (\mathcal{E}_{sp}); \tilde{W}_s S_n^{m''} (\mathcal{E}_s^2) Y^{n''} (\mathcal{E}_{sp}) \right]_{s,j} \right. \\
& \left. + (\iota_s + \iota_j) t_{smn} \left[\tilde{W}_s S_n^m (\mathcal{E}_s^2) Y^n (\mathcal{E}_{sp}); \tilde{W}_j S_n^{m''} (\mathcal{E}_s^2) Y^{n''} (\mathcal{E}_{sp}) \right]_{s,j} \right\}, \quad (85)
\end{aligned}$$

with

$s = 1, 2, \dots, \nu$; number of constituent gas,

$m = 0, 1, \dots, \xi-1$; } number of terms of the expansion

$n = 0, 1, \dots, \xi-1$; } series being used.

It can be shown that the only solution to this set of equations together with constraint equation [Eq. (82)] is;

$$t_s = 1. \quad ; \quad s = 1, 2, \dots, \nu \quad (86)$$

Putting this value back into Eq. (85) then the constants t_{smn} can be determined by the equation

$$\sum_j \sum_{\substack{m=0 \\ n=0}}^{\xi-1} Q_{sj}^{mn} t_{jmn}(\xi) = -R_{smn} \quad , \quad (87)$$

with

$$Q_{sj}^{mn} = \sum_q n_s n_q \left\{ \delta_{sj} \left[\tilde{W}_s^m S_n^{\prime} Y^n ; \tilde{W}_s^m S_n^{\prime} Y^n \right]_{sq} \right. \\ \left. + \delta_{jq} \left[\tilde{W}_s^m S_n^{\prime} Y^n ; \tilde{W}_q^m S_n^{\prime} Y^n \right]_{sq} \right\} \quad . \quad (88)$$

Eqs. (87) and (88) together with the solubility condition , Eq. (63) are the equations which are used for determining those kinetic coefficients.

2. The Coefficients of Conductivity of Multicomponent Non-Equilibrium Gas in the First Approximation

In Eq. (59) the coefficients A_1 , A_2 , A_3 , A_4 and A_5 are the coefficients of conductivity of a multicomponent non-equilibrium gas, and its general expressions are given by Eq. (58). To the first approximation, which means by only taking the first term of the double series expansion of Eq. (45), these coefficients can be expressed in terms of kinetic coefficients, thus

$$\begin{aligned}
 A_1 &= 3 n_s e_s \frac{k}{m_s} \left[(1 - \alpha_s) (a_{10} + a_{01}) - \alpha_s a'_{00} \right], \\
 A_2 &= n_s e_s \frac{k}{m_s} \alpha_s T_0 g_{00}, \\
 A_3 &= 2 \frac{n_s e_s^2}{m_s} L_{00}, \\
 A_4 &= \frac{14\sqrt{2}}{3} \frac{e_s}{\sqrt{m_s k T_0}} |\beta_s| M_{00}, \\
 A_5 &= 3 n_s e_s \frac{k}{m_s} T_0 (d_{10} + d_{01})
 \end{aligned} \tag{89}$$

For determining these kinetic coefficients in terms of collisional integrals, we shall employ Eqs. (87) and (88) together with Table I and II, which are given in the previous section.

The equation to be solved for a_{mn} and a'_{mn} is

$$F_s^{i n(0)} F_s \left[\left(\frac{m_s C_s^2}{2k T_0} - \frac{5}{2} \right) - \frac{m_s C_s^2}{2k T_0} \alpha_s + \frac{e_{sp} \bar{C}_s}{k T_0} \right] C_s = n_s \sum_j n_j I_{sj} (A_s + A_j), \tag{60}$$

and in this case the trial functions to the present order of approximation will be

$$t_s = \sum_s a_{10} S_{3/2}^2 (C_s^2) \quad , \quad (89a)$$

$$t_s' = \sum_s a_{01} Y^{(1)}(c_{sp}) \quad , \quad (89b)$$

$$t_s'' = \sum_s a_{00}' \quad . \quad (89c)$$

Taking the scalar product of Eq. (60) by Eq. (89) respectively and integrating it over the whole velocity range, it yields

$$\begin{aligned} 2A_{10}^{10} a_{10} + A_{01}^{10} a_{01} + \frac{\alpha_s}{1-\alpha_s} A_{00}^{10} a_{00}' &= \frac{14}{5} \\ A_{10}^{01} a_{10} + 2A_{01}^{01} a_{01} + \frac{\alpha_s}{1-\alpha_s} A_{00}^{01} a_{00}' &= \frac{3}{2} \frac{c_{s, \text{int}}}{k} \text{int} \quad (90) \\ \frac{1-\alpha_s}{\alpha_s} A_{10}^{00} a_{10} + \frac{1-\alpha_s}{\alpha_s} A_{01}^{00} a_{01} + 2A_{00}^{00} a_{00}' &= \frac{14}{5} \end{aligned}$$

where these A's are collisional integrals, it can be easily evaluated by knowing the interaction mechanism between the gas molecules, and they are given in the following;

$$\begin{aligned} A_{00}^{00} &= -\sum_j 8 \left(\frac{\alpha_j}{\alpha_s} \right) \frac{(m_s m_j)^{\frac{1}{2}}}{m_s + m_j} \Omega_{sj}^{(1)} \quad , \\ A_{00}^{01} &= -\sum_j 8 \left(\frac{\alpha_j}{\alpha_s} \right) \left(\frac{c_{s, \text{int}}}{k} \right) \frac{(m_s m_j)^{\frac{1}{2}}}{m_s + m_j} \Omega_{sj}^{(1)} \quad , \\ A_{10}^{00} &= \sum_j 8 \left(\frac{1-\alpha_j}{1-\alpha_s} \right) \frac{m_s m_j^{\frac{1}{2}}}{(m_s + m_j)^2} \left(\Omega_{sj}^{(1)}(2) - \frac{5}{2} \Omega_{sj}^{(1)}(1) \right) \quad , \\ A_{00}^{10} &= -A_{10}^{00} \quad , \\ A_{01}^{00} &= \sum_j 8 \frac{\alpha_j}{\alpha_s} \left(\frac{c_{j, \text{int}}}{k} \right) \frac{(m_s m_j)^{\frac{1}{2}}}{m_s + m_j} \Omega_{sj}^{(1)} \quad , \\ A_{10}^{01} &= \frac{c_{s, \text{int}}}{k} A_{10}^{00} \quad , \end{aligned} \quad (91)$$

$$\begin{aligned}
A_{10}^{10} &= \sum_j 8 \left(\frac{1-\alpha_s}{1-\alpha_s} \right) \frac{m_j}{m_s+m_j} \left\{ \frac{5}{4} \left[\frac{6m_s^2}{(m_s+m_j)^2} + \frac{5m_j^2}{(m_s+m_j)^2} \right] \Omega_{sj}^{(1)} \right. \\
&\quad \left. - \frac{5m_j^2}{(m_s+m_j)^2} \Omega_{sj}^{(2)} + \frac{m_j^2}{(m_s+m_j)^2} \Omega_{sj}^{(3)} \right. \\
&\quad \left. + \frac{2m_s m_j}{(m_s+m_j)^2} \Omega_{sj}^{(2)} \right\} , \\
A_{01}^{01} &= \sum_j \frac{c_{s,int} + c_{j,int}}{k} \left(\frac{1-\alpha_s}{1-\alpha_s} \right) \frac{(m_s m_j)^{\frac{1}{2}}}{m_s+m_j} \Omega_{sj}^{(1)} , \\
A_{01}^{10} &= - \sum_j \frac{c_{j,int}}{k} \left(\frac{1-\alpha_s}{1-\alpha_s} \right) \frac{m_s^{3/2} m_j^{1/2}}{(m_s+m_j)^2} \left(\Omega_{sj}^{(1)}(2) - \frac{5}{2} \Omega_{sj}^{(1)}(1) \right)
\end{aligned}$$

with

$$\Omega_{sj}^{(l)}(r) \equiv \pi \int_0^\infty e^{-g_{sj}^2} \frac{2r+2}{g_{sj}} \phi_{sj}^{(l)} dg_{sj} , \quad (92a)$$

and

$$\phi_{sj}^l \equiv \int (1 - \cos^l \chi) g_{sj} b db , \quad (92b)$$

where χ denotes the scattering angle and b denotes the impact parameter.

Solving from Eq. (90), we obtain

$$\begin{aligned}
&\left\{ \frac{15}{4} (4A_{01}^{01} A_{00}^{00} - A_{01}^{00} A_{00}^{01}) + \frac{15}{4} \frac{\alpha_s}{1-\alpha_s} (A_{01}^{10} A_{00}^{01} - 2A_{01}^{01} A_{00}^{10}) \right. \\
&\quad \left. + \frac{3}{2} \frac{c_{s,int}}{k} (A_{01}^{00} A_{00}^{10} - 2A_{01}^{10} A_{00}^{00}) \right\} \\
[a_{10}] &= \frac{\quad}{\left\{ 2A_{10}^{10} (4A_{01}^{01} A_{00}^{00} - A_{01}^{00} A_{00}^{01}) + A_{10}^{00} (A_{11}^{10} A_{00}^{01} - 2A_{01}^{01} A_{00}^{10}) \right.} \\
&\quad \left. + A_{10}^{01} (A_{01}^{00} A_{00}^{10} - A_{01}^{10} A_{00}^{00}) \right\} } \quad (93)
\end{aligned}$$

$$[a_{01}] = \frac{\left\{ \frac{3}{2} \frac{c_{s,int}}{k} (4A_{10}^{10} A_{00}^{00} - A_{10}^{00} A_{00}^{10}) + \frac{15}{4} \frac{\alpha_s}{1-\alpha_s} (A_{10}^{01} A_{00}^{10} - 2A_{00}^{01} A_{10}^{10}) + \frac{15}{4} (A_{00}^{01} A_{10}^{00} - 2A_{00}^{00} A_{10}^{01}) \right\}}{\left\{ 2A_{01}^{01} (4A_{10}^{10} A_{00}^{00} - A_{10}^{00} A_{00}^{10}) + A_{01}^{00} (A_{10}^{01} A_{00}^{10} - 2A_{10}^{10} A_{00}^{01}) + A_{01}^{10} (A_{10}^{00} A_{00}^{01} - 2A_{00}^{00} A_{10}^{01}) \right\}} \quad (94)$$

$$[a'_{00}] = \frac{\left\{ \frac{15}{4} (4A_{10}^{10} A_{01}^{01} - A_{10}^{01} A_{01}^{10}) + \frac{15}{4} \left(\frac{1-\alpha_s}{\alpha_s} \right) (A_{10}^{00} A_{01}^{10} - 2A_{10}^{00} A_{01}^{01}) + \frac{3}{2} \frac{c_{s,int}}{k} \frac{1-\alpha_s}{\alpha_s} (A_{10}^{00} A_{01}^{10} - 2A_{01}^{00} A_{10}^{10}) \right\}}{\left\{ 2A_{00}^{00} (4A_{10}^{10} A_{01}^{01} - A_{10}^{01} A_{01}^{10}) + A_{00}^{10} (A_{01}^{00} A_{10}^{01} - 2A_{10}^{00} A_{01}^{01}) + A_{00}^{01} (A_{10}^{00} A_{01}^{10} - 2A_{10}^{10} A_{01}^{00}) \right\}} \quad (95)$$

For determining g_{00} , we shall use the following equation.

$$F_s^{in(s)} F_s \left[\frac{n}{n_s} \alpha_s C_s (\delta_{sh} - \delta_{sh}) \right] = n_s \sum_j n_j I_{sj} (G_s^i + G_j^h + G_s^k + G_s^k),$$

and trial function

$$t_s = C_s g_{00}.$$

We find

$$[g_{00}] = \frac{\frac{3}{2} n (4A_{10}^{10} A_{01}^{01} - A_{01}^{10} A_{10}^{01})}{\text{Det } |A|} \quad (96)$$

where

$$\text{Det } |A| = \left\{ 2A_{00}^{00} (4A_{10}^{10} A_{01}^{01} - A_{01}^{10} A_{10}^{01}) + A_{10}^{00} (A_{01}^{10} A_{00}^{01} - 2A_{00}^{10} A_{01}^{01}) + A_{01}^{00} (A_{10}^{00} A_{00}^{01} - 2A_{10}^{10} A_{00}^{01}) \right\} \quad (97)$$

For L_{00} , the equation to be solved is

$$\frac{2}{3} F_s^{in(o)} F_s \left(\frac{m_s C_s^2}{2kT_0} - \frac{3}{2} \right) \frac{e_s C_s}{kT_0} = n_s \sum_j n_j I_{sj} (L_s + L_j).$$

with the trial function

$$t_s = C_s L_{00}.$$

We find

$$[L_{00}] = \frac{1}{8\pi} \frac{1}{[C_s, C_s]_{sj} + \sum_j \frac{e_j}{e_s} [C_s, C_j]_{sj}}, \quad (98)$$

and

$$\begin{aligned} [C_s, C_s]_{sj} &= 8 \frac{m_j}{m_s + m_j} \Omega_{sj}^{(1)} \quad , \\ [C_s, C_j]_{sj} &= 8 \frac{(m_s m_j)^{1/2}}{(m_s + m_j)} \Omega_{sj}^{(1)}. \end{aligned} \quad (99)$$

For determining M_{00} , the equation which needs to be solved is,

$$F_s^{in(o)} F_s \left[\frac{2}{3} \left(\frac{m_s C_s^2}{2kT_0} - \frac{3}{2} \right) + 1 \right] \frac{\beta_s}{n_s kT_0} = n_s \sum_j n_j I_{sj} (M_s \beta_s + M_j \beta_j),$$

and the trial function for the present order of approximation can be expressed

as

$$t_s = \frac{7}{3} M_{00}.$$

Thus we find

$$[M_{00}] = \frac{1}{|u_s - u_j|} \tau_{sj}^{e1} \quad s \neq j. \quad (100)$$

Finally, to determine the kinetic coefficient d_{00} , and while the equation to be solved is

$$F_s^{in(o)} F_s \left(\frac{m_s C_s^2}{2kT_0} - \frac{5}{2} \right) C_s = n_s \sum_j n_j I_{sj} (ID_s + ID_j) ,$$

with the trial function

$$t_s = \sum_{m,n} d_{mn} S_{3/2}^n (C_s^2) Y^n (e_{sp}) .$$

We find

$$[d_{10}] = \frac{14}{5} \frac{A_{01}^{01}}{A_{10}^{10} A_{01}^{01} - A_{01}^{10} A_{10}^{01}} ,$$

and

$$[d_{01}] = -\frac{15}{4} \frac{A_{10}^{01}}{A_{10}^{10} A_{01}^{01} - A_{01}^{10} A_{10}^{01}} .$$

(101)

VII. THE ELECTRIC CONDUCTIVITY AND THE GENERALIZED OHM'S LAW FOR A PARTIALLY IONIZED PLASMA UNDER NON-LTE.

In order to show the significance of these general expressions of the electric conductivity and Ohm's law for a N-components gas mixture under the condition of non-LTE which we present in previous section, we shall choose a model of partially ionized hydrogen plasma, namely, it only consists of electrons, singly ionized ions and neutral atoms.

1. The Generalized Ohm's Law at non-LTE

From Eq. (59) we obtain;

electron current

$$\begin{aligned} J_e = & -n_e e \beta_e + [A_1]_e \frac{\partial T_e}{\partial r} + [A_2]_e d_e + [A_3]_e E \\ & + [A_4]_e (J_{e0} \times H) + [A_5]_e \frac{\partial \alpha_e}{\partial r} \end{aligned} \quad (102)$$

and ion current

$$\begin{aligned} J_i = & n_i e \beta_i - [A_1]_i \frac{\partial T_i}{\partial r} - [A_2]_i d_i - [A_3]_i E \\ & - [A_4]_i (J_{i0} \times H) + [A_5]_i \frac{\partial \alpha_i}{\partial r} \end{aligned} \quad (103)$$

Hence the total current is

$$\begin{aligned} J = J_i + J_e \\ = (n_i \beta_i - n_e \beta_e) e + [A_1]_e \frac{\partial T_e}{\partial r} - [A_1]_i \frac{\partial T_i}{\partial r} \\ + [A_2]_e d_e - [A_2]_i d_i \end{aligned}$$

$$\begin{aligned}
& + \left\{ [A_3]_e - [A_3]_i \right\} IE \\
& + \left\{ [A_4]_e J_{e0} - [A_4]_i J_{i0} \right\} \times |H| + [A_5]_e \frac{\partial \alpha_e}{\partial r} - [A_5]_i \frac{\partial \alpha_i}{\partial r} ,
\end{aligned} \tag{104}$$

We should note that, if the LTE condition can be established, the Eq. (104)

becomes

$$J = \left\{ [A_3]_e - [A_3]_i \right\} IE , \tag{105}$$

this is the ordinary Ohm's law which we used quite often, and if we compute the $[A_3]_e - [A_3]_i$ from Eqs. (89-101) we find this is indeed the result of the electric conductivity given by Spitzer [11] ignoring the higher order terms (e.g., the non-LTE effects).

2. The Electric Conductivity at non-LTE

From Eq. (104) the electric conductivity at non-LTE can be defined

by

$$\sigma_{\text{non-LTE}} = \sigma_{\text{LTE}} + \Delta\sigma , \tag{105}$$

with

$$\begin{aligned}
\sigma_{\text{LTE}} &= [A_3]_e - [A_3]_i \\
&\approx 1.53 \times 10^{-4} \frac{1}{\ln \Lambda} \text{ ohm-cm}
\end{aligned} \tag{106a}$$

$$\ln \Lambda = \ln \left[\frac{3}{z e^3} \left(\frac{k^3 T^3}{\pi n_e} \right)^{\frac{1}{2}} \right] . \tag{106b}$$

$$\begin{aligned}
\Delta \sigma = \frac{1}{EI} \left\{ (n_1 \beta_1 - n_0 \beta_0) e + [A_1]_0 \frac{\partial T_0}{\partial r} - [A_1]_1 \frac{\partial T_1}{\partial r} \right. \\
+ [A_2]_0 dl_0 - [A_2]_1 dl_1 \\
+ \left([A_4]_0 J_{b0} - [A_4]_1 J_{b1} \right) \times |H| \\
\left. + [A_5]_0 \frac{\partial \alpha_0}{\partial r} - [A_5]_1 \frac{\partial \alpha_1}{\partial r} \right\} , \quad (107)
\end{aligned}$$

and those kinetic coefficients $[A_1] \dots$ etc. can be computed from Eqs. (89-101)

and it could be found in the appendix.

VIII. DISCUSSIONS

On the basis of this derivation, we have developed a generalized Ohm's law for multicomponent gas mixture under the condition of non-LTE to a second order approximation, that includes all possible distant collision effects (i.e., both elastic and non-elastic collision), the effects of temperature and pressure gradients of individual component gas, and also the effects of the magnetic field.

The general expressions for the coefficients of Ohm's law in terms collision parameters are presented, and also have developed a simplified expression for these coefficients in a specified model, namely, the hydrogen plasma.

In order to conclude this study, we would like to point out while the effects of non-LTE need to be taken into account, this can be characterized by the number density of electron (n_e) and temperature (T) of the gas by checking with certain particular parameters. For instance:

(1) The steady state and homogeneous plasma

In a steady state and homogeneous plasma of sufficiently limited optical depth such that the photoexcitation and ionization in the plasma (i.e., the reabsorption of radiation) can be neglected, the validity of the condition of LTE can be expected only if collisional rate processes dominate the radiative processes (i.e., the radiative decay and recombination), and thus the criteria for the validity of LTE can be established by taking the ratio of collisional rate and radiative rate, according to the results of Griem [12], it yields

$$n_e \geq 9 \times 10^{17} \left(\frac{E_a^{z-1}}{Z^2 E_I} \right)^3 \left(\frac{kT}{Z^2 E_I} \right)^{\frac{1}{2}}, \quad (108)$$

where n_e , E_a^{z-1} , E_I , Z , and T represent the number density of electron per cm^3 , first excited energy in ev., ionization energy in ev. atomic number and temperature respectively.

This formula implies that for a specified model of gas, at a certain temperature there exist a corresponding electron density for which the LTE can be expected. Such a plotting of n_e vs. T is shown in Fig. 1, and the regions of LTE and non-LTE are indicated for a hydrogen plasma.

(2) The homogeneous transient plasma

In this case, it is usually necessary only to prove that, in addition to fulfillment of the criteria for LTE in a homogeneous and time independent plasma, equilibration times are sufficiently short for the establishment of a quasi-stationary near-LTE state, physically, this is the situation of the gas which is produced in the shock tube behind the reflected shock front. In such plasmas, collisional ionization occurs mainly via excitation into intermediate states. Thus the ionization and excitation relaxation time is determined by the slowest processes in the chain, thus the longest time to establish LTE (i.e., the inverse collisional-excitation rate of the ground state) can be expressed by

$$\tau_1^{z-1,a} \approx \frac{1.1 \times 10^7 Z^3 n_a^z}{f_{21} n_e (n_a^z + n_a^{z-1})} \frac{E_2^{z-1,a}}{Z^2 E_I} \left(\frac{kT}{Z^2 E_I} \right)^{\frac{1}{2}} \quad (109)$$

$$\exp \left(\frac{E_2^{z-1,a}}{kT} \right) \text{ sec.}$$

The validity criteria for LTE then can be established by the inequality

$$\frac{\tau_1^{z-1,a}}{\tau_{\text{char.}}} \begin{matrix} < \\ = \\ > \end{matrix} 1, \quad \begin{matrix} \text{LTE} \\ \text{uncertain} \\ \text{Non-LTE} \end{matrix} \quad (110)$$

with $\tau_{\text{char.}}$ indicates the characteristic time. A case of hydrogen plasma is plotting in Fig. 2.

(3) Validity criteria LTE in steady state gasdynamics

High temperature ionized (or dissociated) gases produced in a channel are stationary for all practical purposes, for instance the plasma produced by a stabilized arc, but conditions vary with respect to the distance. In addition to the requirements for LTE in homogeneous and time-independent case, it is therefore necessary that the spatial variations of the electron temperature be small over distances that a given particle can diffuse in times of the order of the equilibration times. This diffuse time can be expressed by the mean free time approximately

$$\tau_f \approx 3 \times 10^9 \left(\frac{E_I}{kT} \right)^{\frac{1}{2}} (n_a + 10n_e)^{-1}, \quad (111)$$

if $\tau_f < \tau_{\text{char}}$. then the condition of LTE can be expected. A similar plotting for hydrogen gas is given in Fig. 3.

Based on these informations, it is possible for us to know whether the effects of non-LTE should be taken into account or not at before hand in general, if it is necessary to use the generalized Ohm's law as we derived here, however, we then should examine each term in Eq. (107). It is very likely, only one or two terms have significant effects. For instance, in the laboratory for a shock tube study, the terms of pressure and temperature gradients are much more important than the others. In the astrophysics problem, we know there is a discrepancy between the measurement and theory for the prediction of the electrical conductivity in the upper photospheric layers, which the author believes that is because the condition of non-LTE prevails in the upper photospheric layers. This will form the later subject of further investigation.

Finally, we should like to recapitulate the main assumptions on which this theory is based on:

- (1) All the restrictions posed on the Boltzmann equation have inherited.
- (2) The interactions between the particles and the external field have only the electric and magnetic field being considered.
- (3) The energy of the translational degrees freedom and the internal degrees of freedom are assumed to be seperable.
- (4) For the translational degrees freedom, the distribution of each com-

ponent gas throughout the particle velocity space is described by the Maxwellian distribution function plus the correction terms due to the gradients of macroscopic quantities for the translational degrees freedom.

(5) For the internal degrees of freedom, the distribution of each component gas throughout the particles energy space is described by the Boltzmann distribution function.

(6) In the present study only the electronic internal states are considered, which is understood that there is no difficulties to include other types of internal states in this theory.

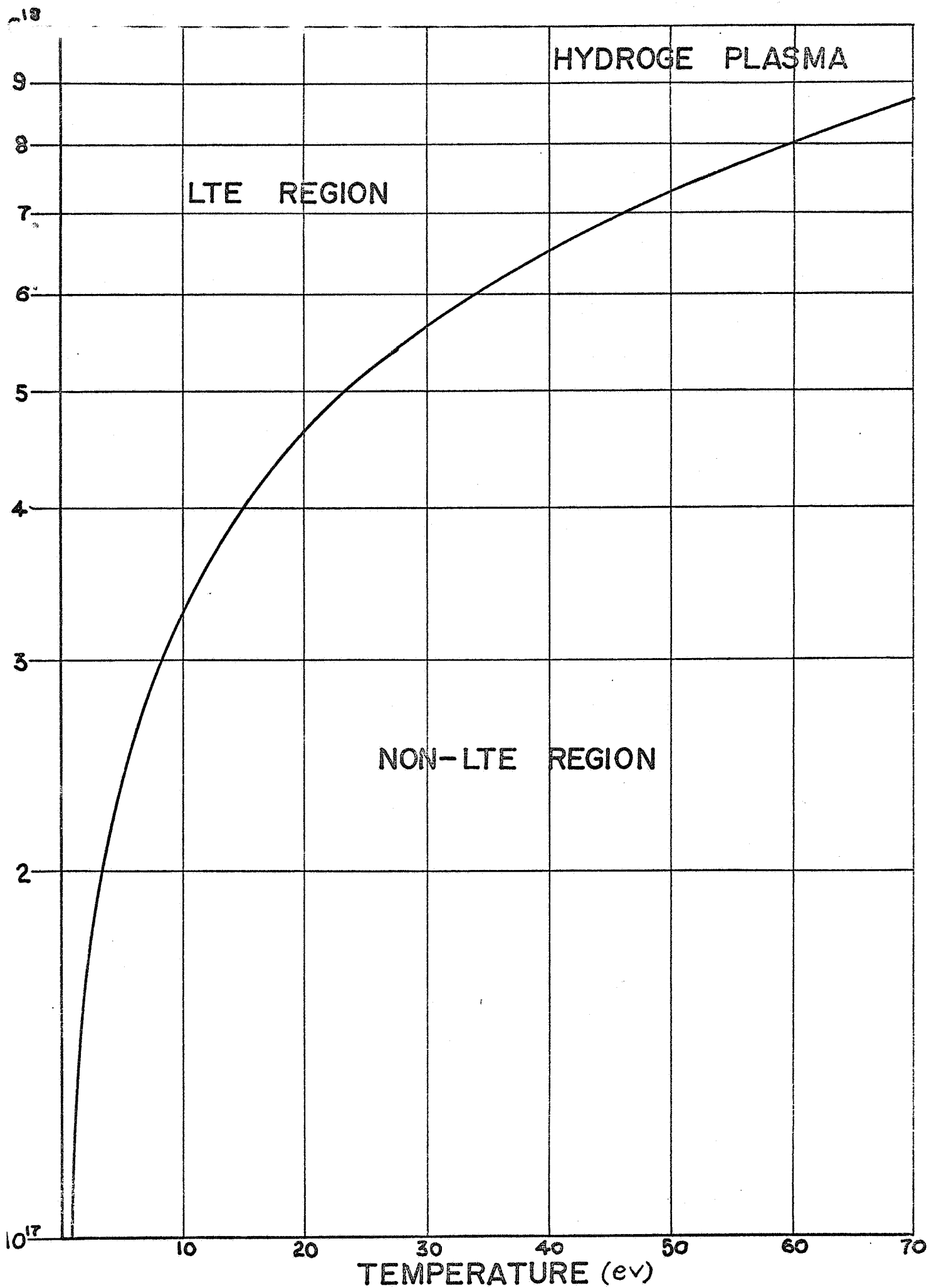
(7) The effects of non-elastic collision are shown explicitly through the collision integrals.

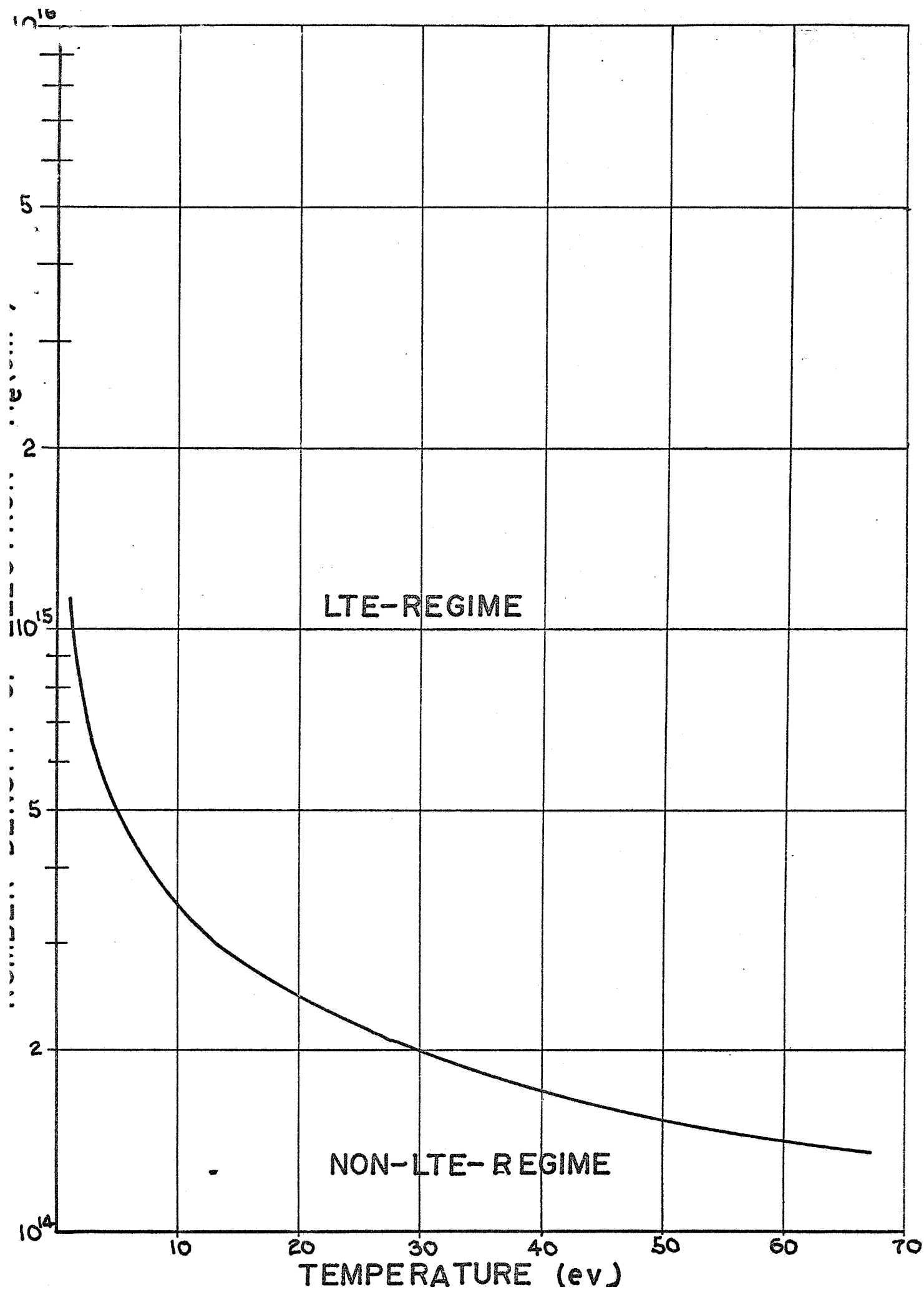
FIGURE CAPTIONS

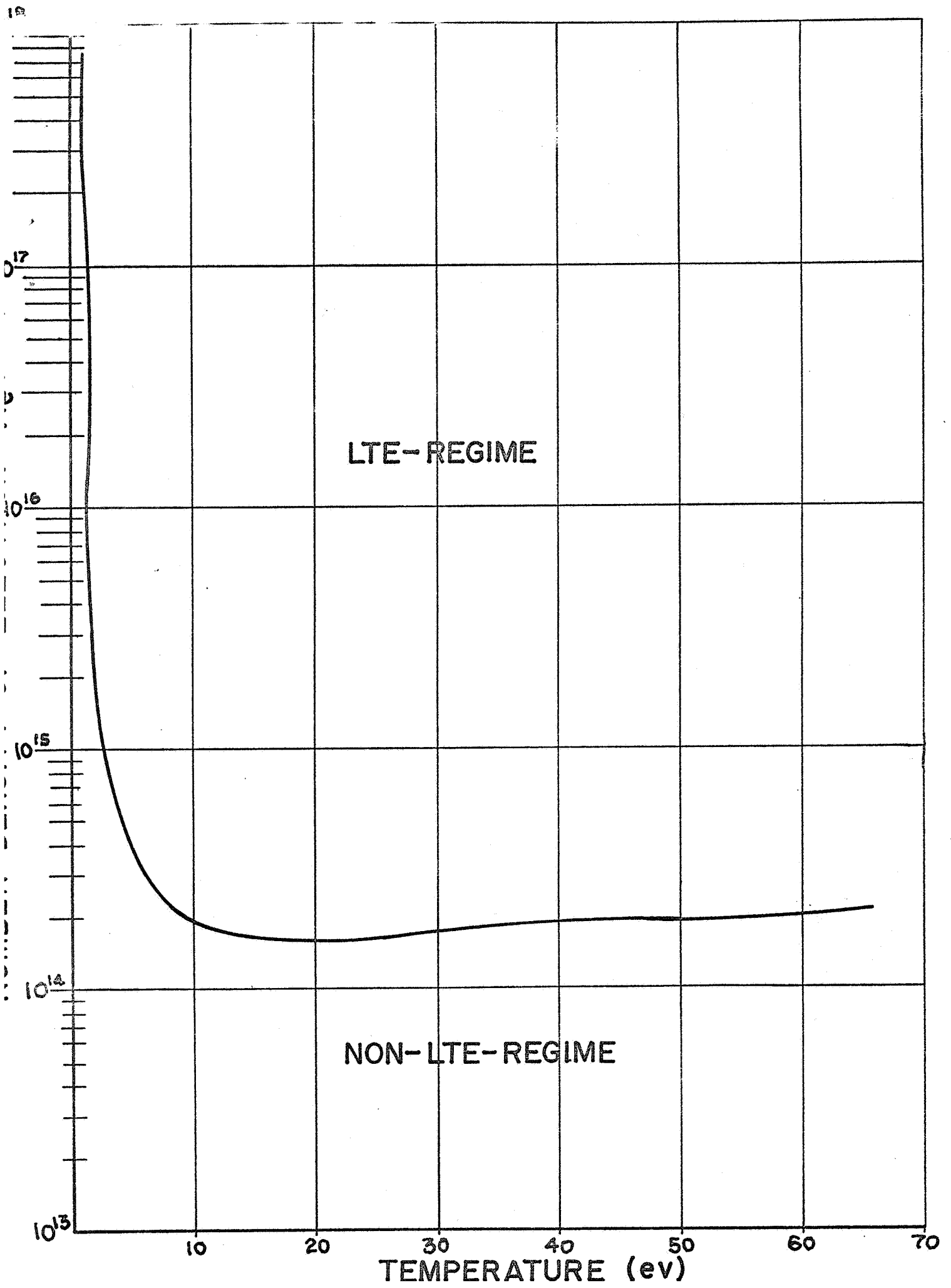
Fig. 1. Electron Number Density vs. Temperature for Steady State and Homogeneous Plasma

Fig. 2. Electron Number Density vs. Temperature for Homogeneous Transient Plasma

Fig. 3. Electron Number Density vs. Temperature for Steady Spatial Inhomogeneous Plasma







BIBLIOGRAPHY:

1. Shi Tsan Wu, Ph.D. Thesis, University of Colorado, 1967 (available from University Microfilms, Inc., Ann Arbor, Michigan).
2. S. Chapman and T. G. Cowling, The Mathematics Theory of Non-Uniform Gases; (Cambridge University Press, England, 1960).
3. S. T. Demetriades and G. S. Argyropoulos, Phys. Fluids 9, 2136 (1966).
4. H. Grad, Comm. Pure and Applied Mathematics 2, 331, (1949).
5. M. Y. Aliyevskiy and V. M. Zhdanov, Zh. Prikl. Mekhoi Tekhn Fiz. No. 5, 11 (1963).
6. Y. Nukagawa: Private communication
7. P. L. Bhatnager, E. P. Gross, and M. Krook, Phys. Rev. 94, #3 (1954).
8. D. Burnett, Proc. London Math Soc., 39, #2048, Ser. 2 (1935).
9. C. S. Wang-Chang, G. E. Uhlenbeck and J. de Boer; J. de Boer and G. E. Uhlenbeck, Editor, "Studies in Statistical Mechanics", II, North Holland Publishing Company, (1964).
10. C. F. Curtis and J. O. Hirschfelder, J. Chem. Phys. 17, #6 (1949).
11. Lyman Spitzer, Jr., Physics of Fully Ionized Gases, Interscience Pub., Inc. New York (1956).
12. Hans R. Griem, "Plasma Spectroscopy", pp. 150-158, McGraw-Hill Book Company, N. Y. (1964).

APPENDIX

To determine the coefficients $A_1, A_2 \dots$ etc., in the Eqs. (102) and (103) for the electron and ion gas, we shall use Eq. (89) together with Eqs. (93-101) by neglecting the terms of order $\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}$ or higher. The following results are obtained;

For electron gas,

$$A_{00}^{00} = -4 \Omega_{ee}^{(1)}(1) ,$$

$$A_{00}^{01} = -4 \frac{c_e \text{ int}}{k} \Omega_{ee}^{(1)}(1) ,$$

$$A_{10}^{00} = 2 \left(\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1) \right) ,$$

$$A_{00}^{10} = -A_{10}^{00} ,$$

$$A_{01}^{00} = 4 \frac{c_e \text{ int}}{k} \Omega_{ee}^{(1)}(1) ,$$

$$A_{10}^{01} = \frac{c_e \text{ int}}{k} A_{10}^{00} ,$$

$$A_{10}^{10} = 4 \left[\frac{55}{16} \Omega_{ee}^{(1)}(1) - \frac{5}{4} \Omega_{ee}^{(1)}(2) + \frac{1}{4} \Omega_{ee}^{(1)}(3) + \frac{1}{2} \Omega_{ee}^{(2)}(2) \right]$$

$$+ 8 \left(\frac{1-\alpha_i}{1-\alpha_e} \right) \left[\frac{25}{4} \Omega_{ei}^{(1)}(1) - 5 \Omega_{ei}^{(1)}(2) + \Omega_{ei}^{(1)}(3) \right] ,$$

(A-1)

$$A_{01}^{01} = \frac{c_e, \text{int}}{k} \Omega_{ee}^{(1)}(1) ,$$

$$A_{01}^{10} = - \frac{1}{4} \frac{c_e, \text{int}}{k} (\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1)) ,$$

$$\begin{aligned} [a_{10}]_e &= \frac{15}{8} \frac{1}{A_{10}^{10}} + \frac{15}{8} \left(\frac{\alpha_e}{1-\alpha_e} \right) \frac{A_{01}^{10} A_{00}^{01} - 2 A_{01}^{01} A_{00}^{10}}{A_{10}^{10} (4 A_{01}^{01} A_{00}^{00} - A_{01}^{00} A_{00}^{01})} \\ &\quad + \frac{3}{4} \frac{c_e, \text{int}}{k} \frac{A_{01}^{00} A_{00}^{10} - 2 A_{01}^{10} A_{00}^{00}}{A_{10}^{10} (4 A_{01}^{01} A_{00}^{00} - A_{01}^{00} A_{00}^{01})} , \end{aligned}$$

$$\begin{aligned} &\frac{15}{8} + \frac{\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1)}{\Omega_{ee}^{(1)}(1) \left(\frac{c_e, \text{int}}{k} - 1 \right)} \left[\frac{5}{128} \left(\frac{\alpha_e}{1-\alpha_e} \right) \left(\frac{c_e, \text{int}}{k} + 4 \right) - \frac{15}{32} \frac{c_e, \text{int}}{k} \right] \\ &= \frac{\left\{ 4 \left[\frac{55}{16} \Omega_{ee}^{(1)}(1) - \frac{5}{4} \Omega_{ee}^{(1)}(2) + \frac{1}{4} \Omega_{ee}^{(1)}(3) + \frac{1}{3} \Omega_{ee}^{(1)}(2) \right] \right.}{\left. + 8 \left(\frac{1-\alpha_e}{1-\alpha_e} \right) \left[\frac{25}{4} \Omega_{ei}^{(1)}(1) - 5 \Omega_{ei}^{(1)}(2) + \Omega_{ei}^{(1)}(3) \right] \right\}} \end{aligned}$$

(A-2)

$$\begin{aligned}
[a_{01}]_e &= \frac{3}{4} \frac{c_{e, \text{int}}}{k} \frac{1}{A_{01}^{01}} + \frac{15}{8} \frac{\alpha_e}{1-\alpha_e} \frac{A_{10}^{01} A_{00}^{10} - 2 A_{00}^{10} A_{10}^{10}}{A_{01}^{01} (4 A_{10}^{10} A_{00}^{00} - A_{10}^{00} A_{00}^{10})} \\
&\quad + \frac{15}{8} \frac{A_{00}^{01} A_{10}^{00} - 2 A_{00}^{00} A_{10}^{01}}{A_{01}^{01} (4 A_{10}^{10} A_{00}^{00} - A_{10}^{00} A_{00}^{10})} \\
&= \frac{3}{4} \frac{1}{\Omega_{ee}^{(1)}(1)} + \frac{15}{8} \frac{\left(\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1) \right) \left[\left(\frac{\alpha_e}{1-\alpha_e} \right) \left(\frac{2k}{c_{e, \text{int}}} - 1 \right) \left(\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1) \right) + 2 \right]}{\Omega_{ee}^{(1)}(1) \left\{ -16 \Omega_{ee}^{(1)}(1) \left[\frac{55}{16} \Omega_{ee}^{(1)}(1) - \frac{5}{2} \Omega_{ee}^{(1)}(2) + \frac{1}{4} \Omega_{ee}^{(1)}(3) \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \Omega_{ee}^{(2)}(2) + \right. \right. \\
&\quad \left. \left. + 2 \frac{1-\alpha_1}{1-\alpha_e} \left(\frac{25}{4} \Omega_{e1}^{(1)}(1) - 5 \Omega_{e1}^{(1)}(2) + \Omega_{e1}^{(2)}(2) \right) \right] \right. \\
&\quad \left. + \left(\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1) \right)^2 \right\}}
\end{aligned} \tag{A-3}$$

$$\begin{aligned}
[a_{00}]_e &= \frac{15}{8} \frac{1}{A_{00}^{00}} + \frac{15}{8} \frac{1-\alpha_e}{\alpha_e} \frac{A_{10}^{00} A_{10}^{01} - 2 A_{10}^{00} A_{01}^{01}}{A_{00}^{00} (4 A_{10}^{10} A_{01}^{01} - A_{10}^{01} A_{01}^{10})} \\
&\quad + \frac{3}{4} \frac{c_{e, \text{int}}}{k} \frac{1-\alpha_e}{\alpha_e} \frac{A_{10}^{00} A_{01}^{10} - 2 A_{01}^{00} A_{10}^{10}}{A_{00}^{00} (4 A_{10}^{10} A_{01}^{01} - A_{10}^{01} A_{01}^{10})}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \begin{aligned} & -\frac{15}{8} (\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1)) (\Omega_{ee}^{(1)}(2) - \frac{7}{2} \Omega_{ee}^{(1)}(1)) \\ & + \frac{3}{4} \frac{c_{e,int}}{k} \left[\frac{1}{8} (\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1))^2 \right. \\ & + 4 \Omega_{ee}^{(1)}(1) \left(\frac{55}{16} \Omega_{ee}^{(1)}(1) - \frac{5}{2} \Omega_{ee}^{(1)}(2) + \frac{1}{4} \Omega_{ee}^{(1)}(3) \right. \\ & \left. \left. + \frac{1}{2} \Omega_{ee}^{(2)}(2) \right) + 8 \Omega_{ee}^{(1)}(1) \frac{1-\alpha_1}{1-\alpha_e} \left(\frac{25}{4} \Omega_{e1}^{(1)}(1) \right. \right. \\ & \left. \left. - 5 \Omega_{e1}^{(1)}(2) + \Omega_{e1}^{(1)}(3) \right) \right] \end{aligned} \right\} \\
= & -\frac{15}{32} \frac{1}{\Omega_{ee}^{(1)}(1)} + \frac{1-\alpha_e}{\alpha_e} \left\{ \begin{aligned} & 16 \left[\Omega_{ee}^{(1)}(1) \right]^2 \left[\frac{55}{16} \Omega_{ee}^{(1)}(1) - \frac{5}{4} \Omega_{ee}^{(1)}(2) + \frac{1}{4} \Omega_{ee}^{(1)}(3) \right. \\ & \left. + \frac{1}{2} \Omega_{ee}^{(2)}(2) + 8 \frac{1-\alpha_1}{1-\alpha_e} \left(\frac{25}{4} \Omega_{e1}^{(1)}(1) - 5 \Omega_{e1}^{(1)}(2) + \Omega_{e1}^{(1)}(3) \right) \right] \\ & + \frac{1}{2} \left(\frac{c_{e,int}}{k} \right) \Omega_{ee}^{(1)}(2) \left(\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1) \right)^2 \end{aligned} \right\}, \quad (A-4)
\end{aligned}$$

$$[g_{00}]_e = \frac{3}{4} n \frac{1}{A_{00}^{00}} = - \frac{3}{16} \frac{n}{\Omega_{ee}^{(1)}(1)}, \quad (A-5)$$

$$[L_{00}]_e = \frac{1}{64\pi} \frac{1}{\Omega_{ee}^1(1)}, \quad (A-6)$$

$$[M_{00}]_e = \frac{1}{|u_{1e} - u_{1i}|} \tau_{me1}^{e1} \quad (A-7)$$

$$\begin{aligned} [d_{10}]_e &= \frac{14}{5} \frac{A_{01}^{01}}{A_{10}^{10} A_{01}^{01} - A_{01}^{10} A_{10}^{01}} \\ &= \frac{14}{5} \frac{\Omega_{ee}^1(1)}{4\Omega_{ee}^{(1)}(1) \left[\frac{55}{16} \Omega_{ee}^{(1)}(1) - \frac{5}{2} \Omega_{ee}^{(1)}(2) + \frac{1}{4} \Omega_{ee}^{(1)}(3) + \frac{1}{2} \Omega_{ee}^{(2)}(2) \right.} \\ &\quad \left. + 2 \left(\frac{1-\alpha_1}{1-\alpha_e} \right) \left(\frac{25}{4} \Omega_{e1}^{(1)}(1) - 5 \Omega_{e1}^{(1)}(2) + \Omega_{e1}^{(1)}(3) \right) \right] \\ &\quad + \frac{1}{2} \frac{c_{e,int}}{k} \left(\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1) \right)^2 \end{aligned} \quad (A-8)$$

$$d_{01e} = - \frac{15}{4} \frac{A_{10}^{01}}{A_{10}^{10} A_{01}^{01} - A_{01}^{10} A_{10}^{01}}$$

$$\begin{aligned}
= & - \frac{15}{4} \frac{\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1)}{2 \Omega_{ee}^{(1)}(1) \left[\frac{55}{16} \Omega_{ee}^{(1)}(1) - \frac{5}{2} \Omega_{ee}^{(1)}(2) + \frac{1}{4} \Omega_{ee}^{(1)}(3) + \frac{1}{2} \Omega_{ee}^{(2)}(2) \right.} \\
& \left. + 2 \left(\frac{1-\alpha_i}{1-\alpha_e} \right) \left(\frac{25}{4} \Omega_{ei}^{(1)}(1) - 5 \Omega_{ei}^{(1)}(2) + \Omega_{ei}^{(1)}(3) \right) \right] \\
& + \frac{1}{4} \frac{c_{e, \text{int}}}{k} \left(\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1) \right)^2
\end{aligned} \quad , \quad (\text{A-9})$$

or

$$\frac{[d_{10}]_e}{[d_{01}]_e} = - \frac{28}{75} \frac{\Omega_{ee}^{(1)}(1)}{\Omega_{ee}^{(1)}(2) - \frac{5}{2} \Omega_{ee}^{(1)}(1)} \quad . \quad (\text{A-10})$$

For ion gas,

$$A_{00}^{00} = -4 \Omega_{11}^{(1)}(1) ,$$

$$A_{00}^{01} = -4 \frac{c_{1,int}}{k} \Omega_{11}^{(1)}(1) ,$$

$$A_{10}^{00} = 2 \left(\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1) \right) ,$$

$$A_{00}^{10} = -A_{10}^{00} ,$$

$$A_{01}^{00} = 4 \frac{c_{1,int}}{k} \Omega_{11}^{(1)}(1) ,$$

$$A_{10}^{01} = \frac{c_{1,int}}{k} A_{10}^{00} ,$$

$$A_{10}^{10} = \frac{55}{4} \Omega_{11}^{(1)}(1) - 5 \Omega_{11}^{(1)}(2) + \Omega_{11}^{(1)}(3) + 2 \Omega_{11}^{(2)}(2) ,$$

$$A_{01}^{01} = \frac{c_{1,int}}{k} \Omega_{11}^{(1)}(1) ,$$

$$A_{01}^{10} = -\frac{1}{4} \frac{c_{1,int}}{k} \left(\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1) \right) .$$

(A-11)

$$\begin{aligned} & \frac{15}{8} + \frac{\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1)}{\Omega_{11}^{(1)}(1) \left(\frac{c_{1,int}}{k} - 1 \right)} \left[\frac{1}{16} \frac{\alpha_1}{1-\alpha_1} \left(\frac{c_{1,int}}{k} + 4 \right) - \frac{1}{4} \frac{c_{1,int}}{k} \right] \\ \left[a_{10} \right]_i &= \frac{\frac{15}{8} + \frac{\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1)}{\Omega_{11}^{(1)}(1) \left(\frac{c_{1,int}}{k} - 1 \right)} \left[\frac{1}{16} \frac{\alpha_1}{1-\alpha_1} \left(\frac{c_{1,int}}{k} + 4 \right) - \frac{1}{4} \frac{c_{1,int}}{k} \right]}{\frac{55}{4} \Omega_{11}^{(1)}(1) - 5 \Omega_{11}^{(1)}(2) + \Omega_{11}^{(1)}(3) + 2 \Omega_{11}^{(2)}(2)} , \quad (A-12) \end{aligned}$$

$$\begin{aligned}
[a_{01}]_i &= \frac{3}{4} \frac{1}{\Omega_{11}^{(1)}(1)} + \frac{15}{8} \frac{(\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1)) \left[\frac{\alpha_1}{1-\alpha_1} \left(\frac{2k}{c_{1,int}} - 1 \right) (\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1)) + 2 \right]}{-4 \Omega_{11}^{(1)}(1) \left[\frac{55}{4} \Omega_{11}^{(1)}(1) - 5 \Omega_{11}^{(1)}(2) + \Omega_{11}^{(1)}(3) \right.} \\
&\quad \left. \Omega_{11}^{(1)}(1) + 2 \Omega_{11}^{(2)}(2) \right]} \\
&\quad + (\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1))^2 \quad (A-13)
\end{aligned}$$

$$\begin{aligned}
[a_{00}]_i &= -\frac{15}{32} \frac{1}{\Omega_{11}^{(1)}(1)} + \frac{1-\alpha_1}{\alpha_1} \frac{\frac{15}{8} (\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1)) (\Omega_{11}^{(1)}(2) - \Omega_{11}^{(1)}(1))}{-4 \Omega_{11}^{(1)}(1) \left[\frac{55}{4} \Omega_{11}^{(1)}(1) - 5 \Omega_{11}^{(1)}(2) + \Omega_{11}^{(1)}(3) \right.} \\
&\quad \left. + \frac{3}{4} \frac{c_{1,int}}{k} \Omega_{11}^{(1)}(1) \left[\frac{55}{4} \Omega_{11}^{(1)}(1) - 5 \Omega_{11}^{(1)}(2) + \Omega_{11}^{(1)}(3) + 2 \Omega_{11}^{(1)}(2) \right] \right]} \\
&\quad + \frac{1}{2} \frac{c_{1,int}}{k} (\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1))^2 \quad (A-14)
\end{aligned}$$

$$[g_{00}]_i = -\frac{3}{16} \frac{n}{\Omega_{11}^{(1)}(1)} \quad (A-15)$$

$$[L_{00}]_i = \frac{1}{64\pi} \frac{1}{\Omega_{11}^{(1)}(1)} \quad (A-16)$$

$$[M_{00}]_i = \frac{1}{|u_{11} - u_{10}|} \tau_{m_{10}}^{el} \quad (A-17)$$

$$\begin{aligned}
[d_{10}]_i &= \frac{14}{5} \frac{\Omega_{11}^{(1)}(1)}{\Omega_{11}^{(1)}(1) \left[\frac{55}{4} \Omega_{11}^{(1)}(1) - 5 \Omega_{11}^{(1)}(2) + \Omega_{11}^{(1)}(3) + 2 \Omega_{11}^{(2)}(2) \right]} \quad (A-18) \\
&\quad + \frac{1}{2} \frac{c_{1,int}}{k} (\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1))^2
\end{aligned}$$

$$\begin{aligned}
 [d_{01}]_i = & -\frac{15}{2} \frac{\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1)}{\Omega_{11}^{(1)}(1) \left[\frac{55}{4} \Omega_{11}^{(1)}(1) - 5 \Omega_{11}^{(1)}(2) + \Omega_{11}^{(1)}(3) + 2 \Omega_{11}^{(2)}(2) \right]} \\
 & + \frac{1}{2} \frac{c_{1, \text{int}}}{k} \left(\Omega_{11}^{(1)}(2) - \frac{5}{2} \Omega_{11}^{(1)}(1) \right)^2 \quad \cdot \quad (\text{A-19})
 \end{aligned}$$